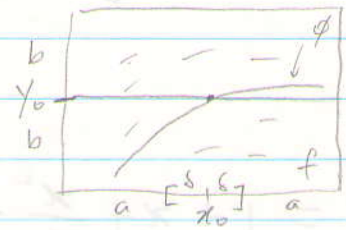
Theorem

Let $f: R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \longrightarrow \mathbb{R}$ be cont. & uniformly Lipschitz in y. $\exists K$ s.t. $|f(x_1, y_1) - f(x, y_2)| < K|y_1 - y_2|$ then the equation $\boxed{\phi' = f(x, \phi)}$, $\phi(x_0) = y_0$ has a unique solution on $[x_0 - \delta, x_0 + \delta]$.

where $\delta = \min(a, \frac{b}{M})$ & $\underline{M}$ is a bound on f on R



proof

Rewrite equation as

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt$$

Idea: Take $\phi_0(x) = y_0$; $\phi_n(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{n-1}(t))\,dt$

example

$y' = y$ $\quad \underset{x_0}{\phi(0)} = \underset{y_0}{1}$

$f(x, y) = y$

$\phi_0(x) = 1$

$$\phi_1(x) = 1 + \int_0^x f(t, \phi_0(t))\,dt$$

$$= 1 + \int_0^x 1 \cdot dt$$

$$= 1 + x$$

$$\phi_2(x) = 1 + \int_0^x f(t, \phi_1(t))\,dt$$

$$= 1 + \int_0^x (1 + t)\,dt$$

$$= 1 + x + \frac{x^2}{2}$$

$$\phi_3(x) = 1 + \int_0^x \left(1 + x + \frac{x^2}{2}\right) dt$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$\vdots$

$$\phi_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

Claim 1

ϕ_n is well defined.

More precisely, ϕ_n is cont. & $\forall x \in [x_0 - \delta, x_0 + \delta]$, $|\phi_n(x) - y_0| \le b$

proof

By induction, (true for ϕ_0). assume true for ϕ_{n-1}.

1. f_n is cont. being the integral of a well-defined cont. function

2. $$|\phi_n(x) - y_0| = \left| \int_{x_0}^x f(t, \phi_{n-1}(t)) dt \right|$$

$$\le \left| \int_{x_0}^x |f(t, \phi_{n-1}(t))| dt \right| \quad (\because |\textstyle\int f| \le |\textstyle\int |f||)$$

$$\le \left| \int_{x_0}^x M dt \right|$$

$$= M |x_0 - x|$$

$$\le M \cdot \delta$$

$$\le M \cdot \frac{b}{M}$$

$$= b$$

Claim 2

For $n \ge 1$, $|\phi_n(x) - \phi_{n-1}(x)| \le \frac{M \cdot K^{n-1}}{n!} |x - x_0|^n$

proof

$$|\phi_n(x) - \phi_{n-1}(x)| = \left| \int_{x_0}^{x} f(t, \phi_{n-1}(t))\,dt - \int_{x_0}^{x} f(t, \phi_{n-2}(t))\,dt \right|$$

$$\le \left| \int_{x_0}^{x_0} \underbrace{|f(t, \phi_{n-1}(t)) - f(t, \phi_{n-2}(t))|}_{} \,dt \right|$$

$$\le \left| \int_{x_0}^{x} K |\phi_{n-1}(t) - \phi_{n-2}(t)|\,dt \right| \text{ (use induction)}$$

$$\le \left| \int_{x_0}^{x} K \frac{M \cdot K^{n-2}}{(n-1)!} \cdot |t - x_0|^{n-1}\,dt \right|$$

$$= \frac{M \cdot K^{n-1}}{(n-1)!} \int_0^{x - x_0} t^{n-1}\,dt$$

$$= \frac{M \cdot K^{n-1}}{(n-1)!} \cdot \frac{|x - x_0|^n}{n}$$

$$= \frac{M \cdot K^{n-1}}{n!} |x - x_0|^n$$

$$\Rightarrow |\phi_n(x) - \phi_{n-1}(x)| \le \frac{M K^{n-1}}{n!} \delta^n$$

$$= C_n$$

where $C_n \ge 0$, $\sum C_n < \infty$

Claim

If $\phi_n(x)$ is a sequence of functions such that

$|\phi_n(x) - \phi_{n-1}(x)| \le C_n$; $\sum C_n < \infty$

Then ϕ_n converges uniformly to some function ϕ.

$$\phi(x) = \lim_{n \to \infty} \phi_n(x)$$

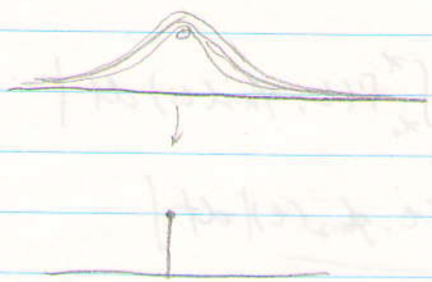
$\forall x, \forall \varepsilon\ \exists N.\ n > N\ |\phi_n(x) - \phi(x)| < \varepsilon$ ← convergence

$\forall \varepsilon\ \exists N\ \forall x.\ n > N\ |\phi_n(x) - \phi(x)| < \varepsilon$ ← uniform convergence.

Moral

$\phi_n \to \phi$ uniformly

1. If $\phi_n \to \phi$ uniformly & ϕ_n are cont. then ϕ is cont.



2. If $\phi_n \to \phi$ uniformly & ϕ are integrable, then $\int \phi_n \to \int \phi$