Regular Covering Spaces

References:
Bredon, Glen E. Topology and Geometry, 1993.
Massey, William S. A Basic Course in Algebraic Topology, 1991.

The purpose of the following entry is to expand of a very pretty topic that was briefly mentioned in lecture.

Definition. Let (X, x) be a pointed space. A covering space $(\widetilde{X}, p)$ of X is called *regular* if $p_*\pi(\widetilde{X}, \tilde{x})$ is a normal subgroup of $\pi(X, x)$.

A basic fact that one must begin with when studying covering spaces is the following.

Theorem. Suppose that X is an arcwise connected and locally arcwise connected topological space. Then a covering space $(\widetilde{X}, p)$ of X is regular if and only if the group $A(\widetilde{X}, p)$ of automorphisms of $\widetilde{X}$ acts transitively on $p^{-1}(x)$, where $x \in X$ is any point and $p^{-1}(x)$ is regarded as a right $\pi(X, x)$-space.

Let us take a moment to recall the definitions used in the above statement. First, the right-action of $\pi(X, x)$ on $p^{-1}(x)$: given $\tilde{x} \in p^{-1}(x)$ and a closed path $f : I \to X$ at the point x, a lemma proven in class informs us that there is a unique path $\tilde{f}$ with initial point $\tilde{x}$ having the property that $p\tilde{f} = f$. So, if $\alpha \in \pi(X, x)$ is a path class having $f : I \to X$ as a representative, we define $\tilde{x} \cdot \alpha$ to be the terminal point of the path $\tilde{f}$. (The definition makes sense since we have a lemma with says that $p\tilde{f} \simeq p\tilde{g}$ implies $\tilde{f} \simeq \tilde{g}$, so that $\tilde{f}$ and $\tilde{g}$ must have the same terminal points.)

If $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ are covering spaces of X, then a continuous map $\phi : \widetilde{X}_1 \to \widetilde{X}_2$ is said to be a *homomorphism* if $p_1 = p_2 \circ \phi$. It is an *isomorphism* is there is a another homomorphism $\psi : \widetilde{X}_2 \to \widetilde{X}_1$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity maps on $\widetilde{X}_1$ and $\widetilde{X}_2$, respectively. An *automorphism* of a covering space is an isomorphism from the covering space to itself.

Our theorem is actually an immediate consequence of the following two facts, which we'll prove for the reader's convenience. From now on, to avoid overusing certain phrases, we'll assume that all of our spaces are arcwise connected and locally arcwise connected.

Theorem. Suppose $(\widetilde{X}, p)$ is a covering space of (X, x). As $\tilde{x}$ runs over $p^{-1}(x)$, the subgroups $p_*\pi(\widetilde{X}, \tilde{x})$ of $\pi(X, x)$ precisely constitute a conjugacy class.

Proof. Fix $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$. We'll first show that $p_*\pi(\widetilde{X}, \tilde{x}_1)$ and $p_*\pi(\widetilde{X}, \tilde{x}_2)$ are conjugate in $\pi(X, x)$. Since $\widetilde{X}$ is arcwise connected, we can find a path γ in $\widetilde{X}$ with initial point $\tilde{x}_1$ and terminal point $\tilde{x}_2$. If f is a loop in $\widetilde{X}$ at the point $\tilde{x}_1$, the map $[f] \mapsto [\gamma^{-1} f \gamma]$ thus defines an isomorphism from $\pi(\widetilde{X}, \tilde{x}_1)$ to $\pi(\widetilde{X}, \tilde{x}_2)$. However, since $p\gamma$ is a closed path in X, we see that

$$p_*[\gamma^{-1} f \gamma] = [p\gamma]^{-1}(p_*[f])[p\gamma],$$

which finishes the first direction.

Next we need to show that for any $\tilde{y}_1 \in p^{-1}(x)$ and $\alpha \in \pi(X, x)$, there exists a point $\tilde{y}_2 \in p^{-1}(x)$ such that

$$p_*\pi(\widetilde{X}, \tilde{y}_2) = \alpha^{-1}[p_*\pi(\widetilde{X}, \tilde{y}_1)]\alpha. \qquad (*)$$

Given such a $\tilde{y}_1$, choose a closed path $f : I \to X$ representing α, and let $\tilde{f} : I \to \widetilde{X}$ be a path covering f with initial point $\tilde{y}_1$. Choosing $\tilde{y}_2$ to be the terminal point of $\tilde{f}$, it is evident that $(*)$ holds. $\square$

Theorem. Let $(\widetilde{X}, p)$ be a covering space of X and $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, where $x_0 \in X$. There exists an automorphism $\phi \in A(\widetilde{X}, p)$ such that $\phi(\tilde{x}_1) = \tilde{x}_2$ if and only if $p_*\pi(\widetilde{X}, \tilde{x}_1) = p_*\pi(\widetilde{X}, \tilde{x}_2)$.

Proof. First, suppose that $\phi \in A(\widetilde{X}, p)$ satisfies $\phi(\tilde{x}_1) = \tilde{x}_2$. That ϕ is an automorphism implies $p\phi = p$. The map ϕ is thus a lifting of $p : \pi(\widetilde{X}, \tilde{x}_1) \to X$ through $\pi(\widetilde{X}, \tilde{x}_2)$, so we get $\pi(\widetilde{X}, \tilde{x}_1) \subset \pi(\widetilde{X}, \tilde{x}_2)$ from the lifting criterion. Running a similar argument using ϕ^{-1} gives us the opposite inclusion.

For the other direction, suppose that $p_*\pi(\widetilde{X}, \tilde{x}_1) = p_*\pi(\widetilde{X}, \tilde{x}_2)$. The lifting criterion gives us maps $\phi : (\widetilde{X}, \tilde{x}_1) \to (\widetilde{X}, \tilde{x}_2)$ and $\psi : (\widetilde{X}, \tilde{x}_2) \to (\widetilde{X}, \tilde{x}_1)$ such that $p\phi = p$ and $p\psi = p$. Since $\psi\phi(\tilde{x}_1) = \tilde{x}_1$ and $\phi\psi(\tilde{x}_2) = \tilde{x}_2$, it follows from lemma 4.4 in Bredon that $\psi\phi$ and $\phi\psi$ are both identity maps, and hence that ϕ is an automorphism. □

Now, let's apply the theory of regular covering spaces to a problem we all solved a little while ago.

Example. This problem, from homework 9, asked us to find the covering space of the figure-8 space $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$.

To visualise this covering space, consider the octagon as a digraph in which all the edges are directed counter-clockwise, and label the edges alternately a and b. Next, add in another set of edges going between adjacent vertices in the clockwise direction, again alternating between a and b, such that each pair of adjacent edges has only "a-paths" or only "b-paths" joining them. If one draws this picture, it is immediately clear that it corresponds to an 8-sheeted covering space for $S^1 \vee S^1$, with the vertices of the octagon all mapping to the basepoint. For the rest of this example, we'll denote this covering space by $\widetilde{X}$.

Now that we've constructed a covering space for $S^1 \vee S^1$, the rest of the problem amounts to proving that this covering is indeed the correct one. The first thing we do is to show that $\widetilde{X}$ is a *regular* covering space, which reduces, by the theorem, to showing that its automorphism group acts transitively on the fibre $p^{-1}(x_0)$ of the basepoint, which is just the set of vertices of the octagon we started with above. But the following symmetries of the octagon (i.e. elements of D_8, the dihedral group of order 16) all correspond to automorphisms of $\widetilde{X}$:

1. Rotations through any multiple of $\pi/2$ radians;

2. Reflections through any line having angle a multiple of $\pi/4$ radians with the horizontal axis.

Once again, these claims are most easily seen to hold by examining a picture of $\widetilde{X}$. In order to assist the reader in understanding this presentation of automorphisms of $\widetilde{X}$, we remark that many (half, actually) of the elements of D_8 actually fail to correspond to automorphisms. For instance, a counter-clockwise rotation through $\pi/4$ radians would take "a-paths" to "b-paths" and vice versa, which automorphisms are not allowed to do.

However, the subgroup of D_8 generated by the elements in the numbered list above is big enough that each vertex of the octagon can be carried to any other vertex via some isometry in the subgroup, which is to say that $A(\widetilde{X}, p)$ acts transitively on the fibre of the basepoint of X.

The rest of the solution is almost completely algebraic. From group theory, we recall that

$$\langle a, b \mid a^2 = b^2 = (ab)^4 = e \rangle \approx D_4,$$

which has order 8. We now have everything we need to finish the argument.

Let N be the normal subgroup of $\pi(X)$ generated by a^2, b^2, and $(ab)^4$. Since each of these elements corresponds to a loop in $\widetilde{X}$, the subgroup they generate is contained in $p_*\pi(\widetilde{X})$. Moreover, since $p_*\pi(\widetilde{X})$ is normal in $\pi(X)$, we have $N < p_*(\widetilde{X})$. (By definition, N is the *smallest* normal subgroup of $\langle a, b \rangle$ containing the generators.) Thus,

$$8 = [\pi(X) : N] = [\pi(X) : p_*\pi(\widetilde{X})][p_*\pi(\widetilde{X}) : N] = 8[p_*\pi(\widetilde{X}) : N],$$

where the first equality holds since $[\pi(X) : N]$ is just the order of D_4, the last equality holds since $\widetilde{X}$ is an 8-sheeted covering space of X, and the middle inequality comes from using the index theorem of group theory. This shows $[p_*\pi(\widetilde{X}) : N] = 1$, which is to say that $p_*\pi(\widetilde{X}) = N$, completing the proof. $\square$