Thm (Inv. FT)

$f : \mathbb{R}^n \supseteq \mathbb{R}^n$ is C^r $(r \geq 1)$ near a, $\exists Df(a)^{-1}$

$\Rightarrow$ $\exists$ nbhds $U \ni a$, $V \ni b = f(a)$ s.t. $\exists (f|_U)^{-1} : V \longrightarrow U$. it is C^r

TL: $\forall \varepsilon > 0$ $\exists$ nbhd J_ε of a s.t. $\forall x, y \in J_\varepsilon$

$$| f(y) - f(x) - (y - x) | \leq \varepsilon \| y - x \|$$

$$\leq \frac{\varepsilon}{1-\varepsilon} \| f(y) - f(x) \|$$

WLOG, $Df(a) = I$, $a = b = 0$

Done. $V = 0.4 J_{0.1}$

$U = f^{-1}(V)$

$(f|_U)^{-1}$ exists & is cts.

Part IV: f^{-1} is diff at $a = 0$

pre-pf: $f^{-1}(0 + x) = f^{-1}(0) + (Df^{-1})(0)x$ + very small

$f^{-1}(x) = 0 + I \cdot x$ + very small

In TL. take $y = 0$. get

$$| - f(x) + x | \leq \varepsilon | - f(x) |$$ on J_ε

meaning

take $y = f(x)$ so $x = f^{-1}(y)$ on a suff. small nbhd of 0

$$| - y + f^{-1}(y) | \leq \varepsilon | y |$$

So $\dfrac{\| f^{-1}(y) - Iy \|}{\| y \|} < \varepsilon$ near 0.

So as $y \longrightarrow 0$ get

$$\frac{\| f^{-1}(y) - f^{-1}(0) - Iy \|}{\| y \|} .$$

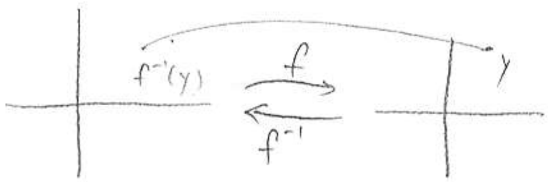
Part V:

f^{-1} is diff near a

repeat argument w/ other pt

Part VI: f^{-1} is C^r near a

pf: $Df^{-1}(y) = (Df(f^{-1}(y)))^{-1}$



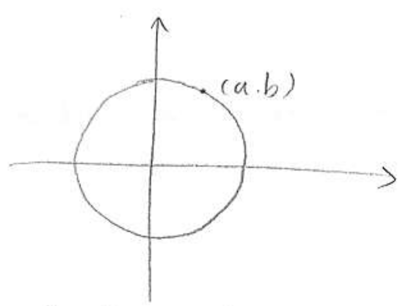
If f is C^2 then f^{-1} is C^1

$(Df^{-1})(y) = Df(f^{-1}(y))^{-1}$

$= \text{explicit formulas involving } \underbrace{\frac{\partial f_i}{\partial f_j}}_{C^1} \ \& \ \underbrace{f^{-1}}_{C^1}$

Can differentiate RHS explicitly. find that all entries of $Df^{-1}(y)$ are differentiable. so f^{-1} is C^2

Thm: (Implicit Function Theorem)



$y = g(x) = \pm\sqrt{1-x^2}$ or none if $x>1$

More precisely near a solution, under good condition, there are more solutions.

$x^2+y^2-1 = f(x,y) = 0 \implies \exists\, g(x)$ s.t. $f(x, g(x)) = 0$

$x^2+y^2=1$

<u>Thm</u>: (Imp. FT)

Given a C^r function, $f : \mathbb{R}^n_{x_1,\dots,x_n} \times \mathbb{R}^k_{y_1,\dots,y_k} \longrightarrow \mathbb{R}^k$

and $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ s.t. $f(a,b) = 0$ & ....

then there exists a unique C^r

$$g : \begin{pmatrix} \text{nbhd } U \\ \text{of } a \end{pmatrix} \longrightarrow \begin{pmatrix} \text{nbhd} \\ \text{of } b \end{pmatrix} \quad \text{s.t. } g(a) = b$$

& $\forall x \in U$. $f(x, g(x)) = 0$. also $Dg = \dots$