Problem Set 6 — MAT257

November 9, 2016

Problems marked with $*$ are to be sumbitted for credit.

1 Munkres §9 (pp.78-79)

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be of class $\mathcal{C}^1$; write f in the form $f(x, y_1, y_2)$. Assume that $f(3, -1, 2) = \mathbf{0}$ and

$$Df(3,-1,2) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

   (a) Show there is a function $g : B \to \mathbb{R}^2$ of class $\mathcal{C}^1$ defined on an open set B in $\mathbb{R}$ such that

$$f\big(x, g_1(x), g_2(x)\big) = \mathbf{0}$$

   for $x \in B$ and $g(3) = (-1, 2)$.

   (b) Find $Dg(3)$.

   (c) Discuss the problem of solving the equation $f(x, y_1, y_2) = \mathbf{0}$ for an arbitrary pair of the unknowns in terms of the third, near the point $(3, -1, 2)$.

$*$ 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of class $\mathcal{C}^1$ with $f(2, -1) = -1$. Set

$$G(x,y,u) = f(x,y) + u^2$$
$$H(x,y,u) = ux + 3y^3 + u^3.$$

The equations $G(x, y, u) = 0$ and $H(x, y, u) = 0$ have the solution $(x, y, u) = (2, -1, 1)$.

   (a) What conditions on Df ensure that there are $\mathcal{C}^1$ functions $x = g(y)$ and $u = h(y)$ defined on an open set in $\mathbb{R}$ that satisfy both equations, such that $g(-1) = 2$ and $h(-1) = 1$?

   (b) Under the conditions of (a), and assuming that $Df(2, -1) = \begin{pmatrix} 1 & -3 \end{pmatrix}$, find $g'(-1)$ and $h'(-1)$.

5. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be of class $\mathcal{C}^1$. "In general", one expects that each of the equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$ represents a smooth surface in $\mathbb{R}^3$, and that their intersection is a smooth curve. Suppose

   i (x_0, y_0, z_0) satisfies both equations

   and

   ii $\partial(f, g)/\partial(x, y, z)$ has rank 2 at (x_0, y_0, z_0).

Then near (x_0, y_0, z_0), one can solve these equations for two of x, y, z in terms of the third, thus representing the solution set locally as a parametrized curve.

$*$ 6. Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class $\mathcal{C}^1$; suppose that $f(\mathbf{a}) = \mathbf{0}$ and that $Df(\mathbf{a})$ has rank n. Show that if $\mathbf{c}$ is a point of $\mathbb{R}^n$ sufficiently close to $\mathbf{0}$, then the equation $f(\mathbf{x}) = \mathbf{c}$ has a solution.

2 Munkres §10 (p.90)

∗ 1. Let $f, g : Q \to \mathbb{R}$ be bounded functions such that $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in Q$. Show that $\underline{\int_Q} f \leq \underline{\int_Q} g$ and $\overline{\int_Q} f \leq \overline{\int_Q} g$.

∗ 3. Let $[0,1]^2 = [0,1] \times [0,1]$. Let $f : [0,1]^2 \to \mathbb{R}$ be given by setting $f(x,y) = 0$ if $y \neq x$, and $f(x,y) = 1$ if $y = x$. Show that f is integrable over $[0,1]^2$.

4. We say $f : [0,1] \to \mathbb{R}$ is **increasing** if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$. If $f, g : [0,1] \to \mathbb{R}$ are increasing and non-negative, show that the function $h(x,y) = f(x)g(y)$ is integrable over $[0,1]^2$.

3 "In addition..."

∗ A. Find an example of a bounded continuous function on the semi-open interval $(0,1]$ which is not uniformly continuous.

B. Prove that every unbounded continuous function on the semi-open interval $(0,1]$ is not uniformly continuous.