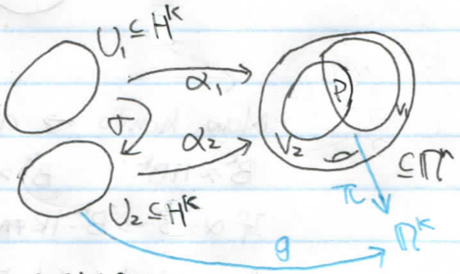
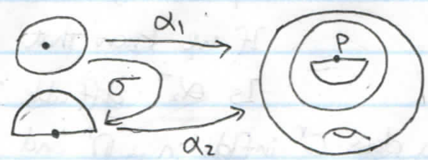
Pf: By def of a "patch" $d\alpha(q)$ has rank k, $n\left\{\overbrace{\begin{pmatrix} d\alpha(q) \end{pmatrix}}^{k}\right.$ so it has k indep. rows.
For convinences, assume these are the first k rows, and let $\pi: \mathbb{R}^n \to \mathbb{R}^k$ be given by dropping the last $(n-k)$ coords of a vector in $\mathbb{R}^n$. $d\pi = \overbrace{\left(\begin{array}{c|c} I_k & 0 \end{array}\right)}^{n}\Big\}k$
Set $g = \pi\alpha$, $dg = (\underset{k}{I} \mid \underset{n-k}{0})(d\alpha) = (\text{first } k \text{ rows of } d\alpha)$
So $dg(q)$ has k indep rows. so $dg(q)$ is invertible.
By the IFT, g is a diffso near q.

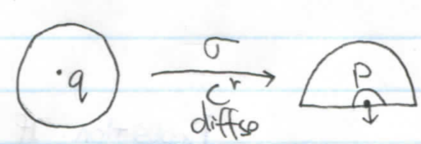
Corollary: "Trasition function are C^r"



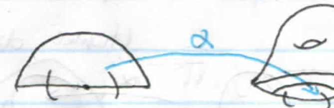
$\sigma = \alpha_2^{-1} \circ \alpha_1$ on $\alpha_1^{-1}(V_1 \cap V_2)$.
Claim: σ is C^r
pf: find a projection: $\pi: \mathbb{R}^n \to \mathbb{R}^k$ s.t.
$g = \pi \circ \alpha_2$ is a diff so near some arbry pt. $p \in V_1 \cap V_2$.
Claim: $\sigma = \alpha_2^{-1} \circ \alpha_1 = g^{-1} \circ \pi \circ \alpha_1$ (near p)
pf: (by composing with g. nts $g \circ \alpha_2^{-1} \circ \alpha_1 \overset{?}{=} g \circ g^{-1} \circ \pi \circ \alpha_1$; $g \circ \alpha_2^{-1} \circ \alpha_1 \overset{?}{=} \pi \circ \alpha_1$
Recall $g = \pi \circ \alpha_2$, $\pi \circ \alpha_2 \circ \alpha_1^{-1} \circ \alpha_1 \overset{?}{=} \pi \circ \alpha_1$ ✓
So $\sigma = g^{-1} \circ \pi \circ \alpha_1$ is a composition of C^r functions, hence it is C^r as required.

Cor. ∂M make sense.





Easy. ∂M is itself a mfd, with no bdy. Sketch:



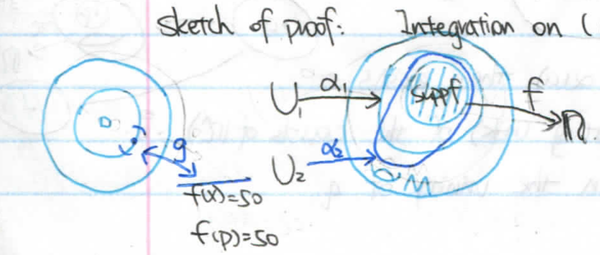
If $f: \mathbb{R}^n \to \mathbb{R}$ is C^r and t is a height at which df has no singularties, then $f^{-1}(t)$ is a mfld.

Jan 30.

Thm. If $A \subseteq \mathbb{R}^n$ is open. $f: A \to \mathbb{R}$ is C^r, then for most h, $f^{-1}(h)$ is a manifold. ($(n-1)$-dim. manifold)
Precisely: If $h \in \mathbb{R}$ s.t. whenever $p \in f^{-1}(h)$, $df(p)$ is of rank 1 ($\Leftrightarrow df(p) \neq 0$)
then $\underbrace{f^{-1}(h)}^{=N}$ is a mfld, so is $M = f^{-1}((-\infty, h])$. Finally, $\partial M = N$.

Example: $f = x^2 + y^2 + z^2$ $df \neq (2x, 2y, 2z)$. So spheres are mflds.

Sketch of proof: Integration on (compact) mflds.

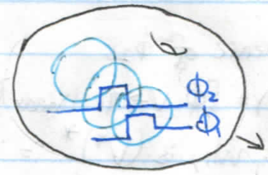


In that case, define $\int_M f\,dV = \int_U (f \circ \alpha) V(d\alpha)$
Q: Does $\int_M f\,dV$, as defined above, depend on α?
$\int_{U_1} (f \circ \alpha_1) V(d\alpha_1) \overset{?}{=} \int_{U_2} (f \circ \alpha_2) V(d\alpha_2)$

What if supp f is big?

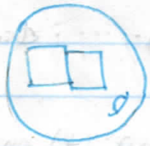
1. Theory answer: use a partition of unity (POI) $\quad \sum \phi_i = 1$

$$\int_M f := \sum_{i=1}^{N} \int_M \phi_i f$$



Thm. This procedure is indep of the POI namely, if ϕ_i & ψ_i are POI, then $\sum \int_M \phi_i f = \sum \int_M \psi_i f$

In practice: Chop to pieces with overlaps of mea-0. integrate on pieces & add.



$v_1, \dots, v_k \mapsto f(v_1 \dots v_k)$ (*) What does f need to be so that we'll be able to use it for an integration theory?

"a diff form" $\int d\omega = \int \omega$

$f(\,\boxed{|||||}\,) = f(\,\boxed{|/|}\,) + f(\,\boxed{/||}\,) \qquad f(v_1 + v_2, v_3) = f(v_1, v_3) + f(v_2, v_3)$

Ans. to (*). 1. f should be linear in each of its arguments.

2. If two of the arguments of f are equal, $f = 0$.

Def: Let V be a v.s. $\mathcal{L}^k(V) = \left\{ f: V^k \to \mathbb{R} : \begin{array}{l} v \mapsto f(v_1, \dots v_{i-1}, v, v_{i+1}, \dots v_k) \\ \text{is linear for every } i,\ 1 \le i \le k \text{ and} \\ \text{every choice of } v_1 \dots v_i \dots v_k \end{array} \right\}$

Challenge: $\dim \mathcal{L}^*(V) = ?$

Feb 01.

Def. $\underbrace{\mathcal{L}^k(V)}_{k\text{-linear forms}} = \{ f: V^k \to \mathbb{R} \mid \forall 1 \le i \le k,\ \forall v_1 \dots v_i \dots v_k \in V,\ v \mapsto f(v_1, \dots v_{i-1}, v, v_{i+1}, \dots v_k) \text{ is linear} \}$

Claim: $\mathcal{L}^k(V)$ is a vector space. better phrasing $\mathcal{L}^k(V)$ is a 'sub-vector space' of $\mathrm{Fun}(V^k \to \mathbb{R})$.

(Aside: X $\quad \mathrm{Fun}(X \to \mathbb{R})$ is a vector space.

N.T.S: if $f, g \in \mathcal{L}^k(V)$ $\quad f+g \in \mathcal{L}^k(V) \qquad$ if $\lambda \in \mathbb{R}$, $f \in \mathcal{L}^k(V)$, $\lambda f \in \mathcal{L}^k(V)$

e.g.
$$\begin{aligned}(f+g)(v+v', v_2, \dots, v_k) &\overset{\text{def of } f+g}{=} f(v+v', v_2, \dots, v_k) + g(v+v', v_2 \dots v_k) \\ &\overset{f,g \in \mathcal{L}^k}{=} f(v, v_2, \dots v_k) + f(v', v_2 \dots v_k) + g(v, v_2 \dots v_k) + g(v', v_2 \dots v_k) \\ &\overset{\text{def of } f+g}{=} (f+g)(v, v_2 \dots v_k) + (f+g)(v', v_2 \dots v_k)\end{aligned}$$

Thm. If $(a_1 \dots a_n)$ is a basis of V and $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k = n^k$

there is a unique $\phi_I \in \mathcal{L}^k(V)$ s.t. $\phi_I(a_{j_1}, \dots, a_{j_k}) = \begin{cases} 1 & \text{if } I = J = (j_1 \dots j_k) \\ 0 & \text{if } I \ne J. \end{cases} = \delta_{IJ}$

Further more: $\{\phi_I : I \in n^k\}$ is a basis of $\mathcal{L}^k(V)$ $\quad (\dim = n^k)$

proof: Step 1: Lemma: an element of $\mathcal{L}^k(V)$ is determined by its values on sequences of basis vectors.

$\Rightarrow$ if $\phi_1, \phi_2 \in \mathcal{L}^k$ have same values on seqs of basis vectors, then $\phi_1 = \phi_2$

So ϕ_I is uniquely determined.

Proof of lemma: Suppose $\phi \in \mathcal{L}^k$ $\quad v_1 = \alpha_{11} a_1 + \alpha_{12} a_2 + \cdots + \alpha_{1n} a_n$

$v_k = \alpha_{k1} a_1 + \cdots \cdots + \alpha_{kn} a_n$

Then $\phi(v_1 \dots v_k) = \phi\left(\sum_{j_1=1}^{n} \alpha_{1j_1} a_{j_1}, \sum_{j_2=1}^{n} \alpha_{2j_2} a_{j_2}, \dots\right) = \sum_{j_1=1}^{n} \alpha_{1j_1} \sum_{j_2=1}^{n} \alpha_{2j_2} \cdots \phi(a_{j_1}, a_{j_2} \cdots a_{j_k})$ $\square$

Step 2. Existence of ϕ_I. V is always f.d.

$\mathcal{L}^1(V) = \{\phi: V\to\mathbb{R} \text{ linear}\} =: V^*$ = "the dual of V"

Aside: why is $(V^*)^*$ "=" V

There is a unique $\phi_i\in V^*$ s.t. $\phi_i(a_j) = \delta_{ij}$ (already known).

If $I=(i_1\dots i_k)$, let $\phi_I(v_1\dots v_k) = \phi_{i_1}(v_1)\cdot\phi_{i_2}(v_2)\cdot\cdots\cdot\phi_{i_k}(v_k)$ $=\phi_{i_1}(a_{j_1})\cdot\phi_{i_2}(a_{j_2})\cdots\phi_{i_k}(a_{j_k})$

a) ϕ_I is multi-linear ... V b) $\phi_I(a_J) = x+y$ $=\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_kj_k} = \delta_{IJ}$

Step 3. $\{\phi_I\}$ is linear indep. Assume (*) $0=\sum_{I\in n^k}\alpha_I\phi_I$. Evaluate (*) on $a_J=(a_{j_1}\dots a_{j_k})$

$0=\sum_I\alpha_I\phi_I(a_J) = \sum_I\delta_{IJ}\alpha_I = \alpha_J$ □.

Step 5. $\{\phi_I\}$ span.

Feb. 03.

V: v.s. with basis $\{a_1,\dots,a_n\}$

Def: $\mathcal{L}^k(V) = \{\text{multi-linear } f: V^k\to\mathbb{R}\}$ "k-tensors", a vector space.

For $I\in n^k$. $\exists!\ \phi_I\in\mathcal{L}^k$ s.t. $\phi_I(a_J) = \phi_I(a_{j_1}\dots a_{j_k}) = \delta_{IJ}$

These make a basis of $\mathcal{L}^k$, so $\dim\mathcal{L}^k(V) = n^k$ Lin. indep: done. NTS: $\{\phi_I\}$ span $\mathcal{L}^k$

Claim: $\{\phi_I\}$ span $\mathcal{L}^k(V)$

pf: Take some $f\in\mathcal{L}^k(V)$ Let $\lambda_I = F(a_I)$.

$(f=\sum_{I\in n^k}\lambda_I\phi_I$ $\lambda_I\in\mathbb{R}$ evaluate a_J $f(a_J) = \sum_I\lambda_I\phi_I(a_J) = \lambda_J)$

Claim: $\sum\lambda_I\phi_I = f$. Enough to show that LHS=RHS when evaluated on a_J's.

In deed. $f(a_J) = \sum_I\lambda_I\phi_I(a_J) = \lambda_J$

tensor product

Extra pt 1. Define $\otimes: \mathcal{L}^k(V)\times\mathcal{L}^m(V)\to\mathcal{L}^{k+m}(V)$.

by if $f\in\mathcal{L}^k$ $g\in\mathcal{L}^m$ $(f\otimes g)(v_1\dots v_{k+n}) = f(v_1\dots v_k)\otimes g(v_{k+1}\dots v_{k+m})$

Already shown that indeed $f\otimes g\in\mathcal{L}^{k+m}(V)$

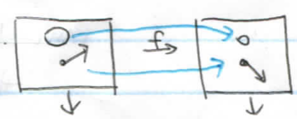
Extra props: 1. $\otimes$ is bilinear: $(f_1+f_2)\otimes g = f_1\otimes g + f_2\otimes g$ λ ...

2. $\otimes$ is assoc: $(f\otimes g)\otimes h = f\otimes(g\otimes h)$

3. $\phi_I(v_1\dots v_k) = \phi_{i_1}(v_1)\cdot\phi_{i_2}(v_2)\cdots\phi_{i_k}(v_k)$ $\phi_I = \phi_{i_1}\otimes\phi_{i_2}\otimes\phi_{i_3}\otimes\cdots\otimes\phi_{i_k}$.

$\mathbb{R}^n\xrightarrow{f}\mathbb{R}^n$ "points push forward" "tangent push forward"

"functions pull back" "subsets pull back"



$f^{-1}(A\cap B) = f^{-1}(A)\cap f^{-1}(B)$.

Given a linear $T: V\to W$.
$$\begin{array}{ccc}\mathcal{L}^k(V) & \xleftarrow{T^*} & \mathcal{L}^k(W)\\ \downarrow & & \downarrow\\ T^*f & & f\end{array}$$

Def. $(T^*f)(v_1\dots v_k) = f(Tv_1\dots Tv_k)$ T^*f really is a mlt of $\mathcal{L}^k(V)$

1: $T^*: \mathcal{L}^k(W)\xrightarrow{f,g}\mathcal{L}^k(V)$ is linear. $\lambda\in\mathbb{R}$. $T^*(f+\lambda g) = (T^*f)+\lambda(T^*g)$.

2. "Contravariant" (all pull pack operations are "contravariant")

$$\begin{tikzcd} V \arrow[r, "T"', "\text{linear}"] \arrow[rr, bend right, "S\circ T"'] & W \arrow[r, "S", "\text{linear}"'] & Z \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}^k(V) & \mathcal{L}^k(W) \arrow[l, "T^*"'] & \mathcal{L}^k(Z) \arrow[l, "S^*"'] \arrow[ll, bend left, "(S\circ T)^*"] \end{tikzcd}$$

$(S\circ T)^* = T^*\circ S^*$