Additive identity is $0+0\sqrt{3}=\mathbf{0}$ and multiplicative identity is $1+0\sqrt{3}=\mathbf{1}$.

$\forall x=(a+b\sqrt{3})\in F_1$, $\exists$ an additive inverse of the form $(-a-b\sqrt{3})\in F_1$, and

$\exists$ a multiplicative inverse $(a+b\sqrt{3})^{-1}$ or $\dfrac{a-b\sqrt{3}}{a^2-3b^2} = \dfrac{a}{a^2-3b^2} - \dfrac{b}{a^2-3b^2}\sqrt{3}$

$a,b\in\mathbb{Q} \Rightarrow \dfrac{a}{b} \neq \pm\sqrt{3} \Rightarrow (\dfrac{a}{b})^2 \neq 3 \Rightarrow a^2 \neq 3b^2 \Rightarrow a^2-3b^2 \neq 0$

$\dfrac{a}{a^2-3b^2}\in\mathbb{Q}$, $\dfrac{-b}{a^2-3b^2}\in\mathbb{Q}$, when $a,b\in\mathbb{Q}$

Hence, F_1 is a field. □

3-2. Is the set $F_2 = \{a+b\sqrt{3} : a,b\in\mathbb{Z}\}$ (with the same addition & multiplication) a field?

It is not true that $\forall x=(a+b\sqrt{3})\in F_2$, $\exists$ a multiplicative inverse in F_2.
Counter example: take $x = 4 + 0\sqrt{3} = 4\in F_2$, its multiplicative inverse is $\frac{1}{4}$, of the form, $a+b\sqrt{3}$, where $a=\frac{1}{4}$ and $b=0$, but $a=\frac{1}{4}\notin\mathbb{Z}$, i.e., x's inverse $(\frac{1}{4}+0\sqrt{3})\notin F_2$. Hence, F_2 is **not** a field.

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4. Let $F_4 = \{0, 1, a, b\}$ be a field containing 4 elements. Assume that $1+1=0$. Prove that $b = a^{-1} = a^2 = a+1$.

Proof: By definition of a field
$\Rightarrow$ The sum, product, difference, & quotient of any two elements in F_4 is also an element of F_4 and field elements are distinct elements. $\Rightarrow$

$ab =$	0	then a=0 or b=0	impossible
	a	then b=1 or a=0	impossible
	b	then a=1 or b=0	impossible
	1	then	$b=a^{-1}$
$a^2 =$	0	then a=0	impossible
	1	then $a=a^{-1}=b$? $a=1$ or $a=-1=1$	impossible
	a	then a(a-1)=0, a=0,1	impossible
	b	$a^2=b$	the only choice
$a+1 =$	0	then a=-1=1 since 1+1=0 or 1=-1	impossible
	1	then a=0	impossible
	a	then 1=0	impossible
	b	a+1=b	the only choice

Hence, $b = a^{-1} = a^2 = a+1$. □