$f : \mathbb{R} \longrightarrow \mathbb{R}$

Near $\rightsquigarrow$ $|x-y| < \varepsilon, \delta$

Def: Let X be a set.

A metric on X is a fct $d : X \times X \longrightarrow \mathbb{R}$ s.t.

1. $d(x,y) = d(y,x)$
2. $d(x,y) \geq 0$ & $d(x,y) = 0 \iff x = y$
3. $\forall x, y, z \in X$. $d(x,y) + d(y,z) \geq d(x,z)$ "triangle inequality"

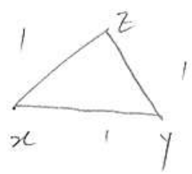
e.g: 0. $X = \mathbb{R}$. $d(x,y) = |x-y|$

1. $X = \mathbb{R}^n$. $d_1(x,y) = \|x-y\| = \left(\sum (x_i - y_i)^2\right)^{1/2}$

$$d_2(x,y) = |x-y| = \max_i |x_i - y_i|$$

2. X any set

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$



3. $X = C([0,1]) = \{$ cts fct $f : [0,1] \longrightarrow \mathbb{R}\}$

$d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$

$f, g \in X$

$f, g : [0,1] \to \mathbb{R}$ cts

Def: A "metric sp" is a set X along with a choice of a metric on it. e.g: $\mathbb{R}$, $\mathbb{R}^n$, $C([0,1])$, ...

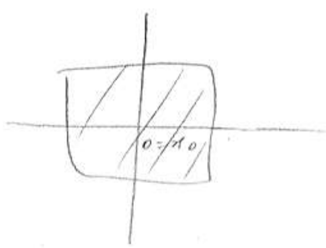
Def: Given a metric sp X & $x_0 \in X$ & $\varepsilon > 0$

$B(x_0, \varepsilon) = \{x \in X \mid d(x_0, x) < \varepsilon\}$

"the ε-nbhd around x_0"

"the ε-ball around x_0"

$\mathbb{R}^2$, $|\cdot|_{sup}$. $B(0,1) =$



$x = (a, b)$

$d(x, 0) = |x - 0|$

$|x| = \max(|a|, |b|)$

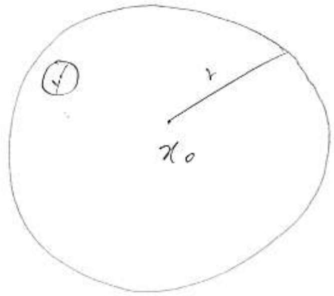
Def: A set $U \subset X$ is called "open" if $\forall x_0 \in U$, $\exists \varepsilon > 0$ s.t.

$B(x_0, \varepsilon) \subset U$

"every pt in U has an ε-nbhd contained in U"

A set $F \subset X$ is closed if $F^c = X \setminus F$ is open

Claim: $B(x_0, r)$ open



pf: Let $y \in B(x_0, r)$

Take $\varepsilon = r - d(x_0, y)$

$\varepsilon > 0$ as $y \in B(x_0, r)$ so $d(x_0, y) < r$.

Claim: $B(y, \varepsilon) \subset B(x_0, r)$

pf: Let $z \in B(y, \varepsilon)$ meaning $d(z, y) < \varepsilon$

Then
$$\begin{aligned} d(x_0, z) &\le d(x_0, y) + d(y, z) \\ &< d(x_0, y) + \varepsilon \\ &= d(x_0, y) + r - d(x_0, y) \\ &= r \end{aligned}$$

So $z \in B(x_0, r)$

$\square$

Thm 1a: 1. $\emptyset$, X are open.

2. An arb. union of opens is open
$\forall \alpha \in I$, U_α open $\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ open

3. $\forall\, 1 \le i \le n$, U_i open $\Rightarrow \bigcap_{i=1}^{n} U_i$