An Application of Euler's Formula in a Combinatorial Problem

Problem (Example 3, Section 7.1, Tucker)

Suppose we draw n straight lines on a piece of paper so that every pair of lines intersect but no three lines are collinear. Into how many regions do these n lines divide the plane?

Solution

$$\frac{n(n+1)}{2}+1$$

Consider the following equivalent proposition of the original problem.

Given a square, draw n line segments $\ell_1, \ell_2, \dots, \ell_n$ such that
(1). every pair of line segments intersect but no three line segments are collinear and
(2). the two endpoints of each ℓ_{i} $(1 \le i \le n)$ lie on two distinct sides of the square.

Into how many regions do these n line segments divide the square?

Let V be the set consisting of the four vertices of the square, the $2n$ endpoints of $\ell_1, \ell_2, \dots, \ell_n$, and the points of intersection of these n line segments. Let G be the set consisting of the line segments whose endpoints are in V. Then $G=(V,E)$ is a connected planar graph, and hence Euler's formula applies, i.e.

$$|V|-|E|+|F|=2,$$

where F is the set of regions in G.

First we find $|V|$. By construction, $\ell_1, \ell_2, \dots, \ell_n$ intersect at

$$\binom{n}{2}=\frac{n(n-1)}{2}$$

points, since each point of intersection lies on exactly two line segments.
Therefore, we have that

$$|V|=\tfrac{n(n-1)}{2}+2n+4.$$

Now we find $|E|$. On the one hand, for each ℓ_{i} $(1 \le i \le n)$, there are $n-1$ points between its two endpoints, dividing it into n edges. Thus the number of edges whose endpoints are both points of intersection formed by $\ell_1, \ell_2, \dots, \ell_n$ is n^2. On the other hand, there are $2n+4$ points on the perimeter of the square, meaning that the perimeter is a cycle of length $2n+4$. So we obtain that

$$|E| = n^2 + 2n + 4.$$

Euler's formula gives that

$$\begin{aligned}
|F| &= |E| - |V| + 2 \\
&= (n^2 + 2n + 4) - \left(\frac{n(n-1)}{2} + 2n + 4\right) + 2 \\
&= n^2 - \frac{n(n-1)}{2} + 2 \\
&= \frac{n(n+1)}{2} + 2
\end{aligned}$$

Observe that the outer region is counted, so the n lines segments divide the square into

$$\frac{n(n+1)}{2} + 2 - 1 = \frac{n(n+1)}{2} + 1$$

regions. Consequently, the n lines in the original problem divide the plane into

$$\frac{n(n+1)}{2} + 1$$

regions. □

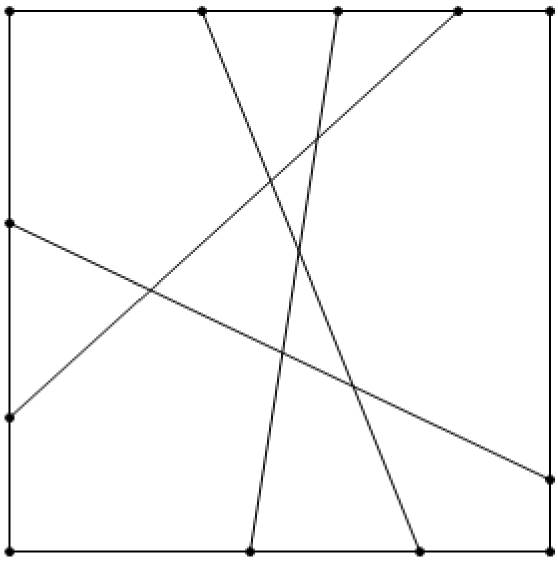

Figure 1. The square is divided into 11 regions by 4 line segments.