HOMEWORK ASSIGNMENT #4

CORRECTION

Algebra I

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Exercise 8 page 41-42: *Let* $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ *be a subset of the vector space* F^3

(a) *Prove that if* $F = \mathbb{R}$, *then* S *is linearly independent*

Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ be a subset of the vector space $\mathbb{R}^3$

Suppose $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$(0,0,0) = a_1(1,1,0) + a_2(1,0,1) + a_3(0,1,1)$$

We obtain the following system of equation:

$$\begin{aligned} a_1 + a_2 &= 0 && e_1 \\ a_1 + a_3 &= 0 && e_2 \\ a_2 + a_3 &= 0 && e_3 \end{aligned} \quad \rightarrow \quad \begin{aligned} 2a_1 &= 0 && e_1 + e_2 - e_3 \\ a_1 + a_3 &= 0 && e_2 \\ a_2 + a_3 &= 0 && e_3 \end{aligned}$$

$$\begin{aligned} a_1 &= 0 && 1/2 \cdot e_1 \\ a_1 + a_3 &= 0 && e_2 \\ a_2 + a_3 &= 0 && e_3 \end{aligned} \quad \rightarrow \quad \begin{aligned} a_1 &= 0 && e_1 \\ a_3 &= 0 && e_2 - e_1 \\ a_2 + a_3 &= 0 && e_3 \end{aligned}$$

$$\begin{aligned} a_1 &= 0 && e_1 \\ a_3 &= 0 && e_2 \\ a_2 &= 0 && e_3 - e_2 \end{aligned}$$

Therefore the only solution for this system is $a_1 = a_2 = a_3 = 0$. Si S is linearly independent.

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(b) *Prove that if* F *has characteristic* 2, *then* S *is linearly dependent*

Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ be a subset of the vector space F^3 where F is a field with characteristic 2.

Since F is a field with characteristic 2 we have that: $1 + 1 = 0$. Therefore if we consider the linear combination of the vectors of S where $a_1, a_2, a_3 \in F$ and $a_1 = a_2 = a_3 = 1$, we get:

$$\begin{aligned} a_1(1,1,0) + a_2(1,0,1) + a_3(0,1,1) &= 1 \cdot (1,1,0) + 1 \cdot (1,0,1) + 1 \cdot (0,1,1) \\ &= (1,1,0) + (1,0,1) + (0,1,1) = (1+1, 1+1, 1+1) = (0,0,0) \end{aligned}$$

Therefore, in a field with characteristic 2, $a_1 = a_2 = a_3 = 1$ is a finite linear combination of vectors of S whose coefficients are not all equals to 0. Hence, S is linearly dependent.

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Exercise 9 page 41: *Let* u *and* v *be distinct vectors in a vector space* V. *Show that* $\{u, v\}$ *is linearly dependent if and only if* u *or* v *is a multiple of the other.*

Let u and v be distinct vectors in a vector space V over a field F.

We show first that u or v is a multiple of the other $\Longrightarrow \{u, v\}$ linearly dependent.

- Suppose u is a multiple of v. Then there exists $q \in F$ such that $u = vq$. From this, we have that: $u - vq = 0_V$. Thus, there exists a non trivial representation of the zero vector as a linear combination of distinct vectors of $\{u, v\}$ since we have $a_1 = 1$.
- Suppose v is a multiple of u. Then there exists $q \in F$ such that $v = uq$. From this, we have that: $v - uq = 0_V$. Thus, there exists a non trivial representation of the zero vector as a linear combination of distinct vectors of $\{u, v\}$ since we have $a_1 = 1$.

Hence u or v is a multiple of the other $\Longrightarrow \{u, v\}$ linearly dependent.

Now let us show that $\{u, v\}$ linearly dependent $\Longrightarrow$ u or v is a multiple of the other

Suppose $\{u, v\}$ linearly dependent. Then there exists $a, b \in F$ such that a, b are not both equal to zero such that $au + bv = 0_V$

- Suppose $a \neq 0$. Then $au + bv = 0_V \Rightarrow au = -bv$. Moreover, since $a \neq 0, \exists a^{-1} \in F$ such that $a \cdot a^{-1} = 1$. Thus, we get: $au = -bv \Rightarrow a^{-1}au = a^{-1}(-bv)$ $\Rightarrow 1 \cdot u = a^{-1}(-bv) \Rightarrow u = (-ba^{-1})v$. Therefore u is a multiple of v.
- Suppose $b \neq 0$. Then $au + bv = 0_V \Rightarrow bv = -au$. Moreover, since $b \neq 0, \exists b^{-1} \in F$ such that $b \cdot b^{-1} = 1$. Thus, we get: $bv = -au \Rightarrow b^{-1}bv = b^{-1}(-au)$ $\Rightarrow 1 \cdot v = b^{-1}(-au) \Rightarrow v = (-ab^{-1})u$. Therefore v is a multiple of u.

Thus $\{u, v\}$ linearly dependent $\Longrightarrow$ u or v is a multiple of the other.
Hence u or v is a multiple of the other $\Longleftrightarrow$ $\{u, v\}$ linearly dependent.

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Exercise 11 page 42: *Let $S = \{u_1, u_2, \dots, u_n\}$ be a linearly independent subset of a vector space V over a field $\mathbb{Z}_2$. How many vectors are there in $Span(S)$? Justify your answer.*

Let $S = \{u_1, u_2, \dots, u_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$.
Let denote by n the number of elements of S which we denote by $|S|$. If $n = 0$, then $S = \emptyset$ (possible since the empty set is linearly independent) then $Span(S) = \{0\}$. Therefore, $|Span(S)| = 1$

Suppose now that $n \geq 1$. We have $S = \{(1)\}$ because $\{(0)\}$ is linearly dependent and any nonempty subset of a vector space that contain 0 is linearly dependent.
Moreover we have $(1) = 1 \cdot (1)$ and $(0) = 0 \cdot (1)$. Thus $Span(S) = \mathbb{Z}_2$ and it has 2 elements.

So the number of elements of the span of a linearly independent subset of a vector space V over a field $\mathbb{Z}_2$ is 1 if the number of elements the subset is 0 and 2 is the number of element of the subset is greater than 1.

General case for a linearly independent subset of a vector space V over a field $\mathbb{Z}_2^k$ with k a positive integer

Let $S = \{u_1, u_2, \dots, u_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2^k$.

We claim that the number of vectors in $Span(S) = 2^n$. We will show it by induction over $|S| = n$.

If $n = 0$, then $S = \emptyset$ (possible since the empty set is linearly independent) then $Span(S) = \{0_V\}$. Therefore, $|Span(S)| = 1 = 2^0$

Base case: We now suppose that $n = 1$. So $S = \{u_1\}$ is non empty. If $v \in span(S)$, there exists $a \in \mathbb{Z}_2$ such that

$$v = au_1$$

Since $\mathbb{Z}_2$ has only two elements we have that $a = 1$ or $a = 0$. Therefore, $span(S) = \{u_1, 0_V\}$ and $|Span(S)| = 2 = 2^1$

Suppose that $|Span(S)| = 2^n$ for any linearly independent subset of a vector space V over $\mathbb{Z}_2^k$ such that $|S| = n \geq 1$. Let us show that it is true for $n + 1$.

Suppose that $S \cup \{u_{n+1}\}$ is linearly independent. Moreover $S \subset S \cup \{u_{n+1}\} \Rightarrow Span(S) \subset Span(S \cup \{u_{n+1}\}) \Rightarrow |Span(S)| = 2^n \leq |Span(S \cup \{u_{n+1}\})|$ since S is finite (and thus so is its span). We want to know how much new vectors does u_{n+1} brings to the span of this new set. Suppose that $v \in Span(S \cup u_{n+1})$. Then

$$v = \sum_{i=1}^{n} a_i u_i + a_{n+1} u_{n+1}$$

For every new vector of $Span(S \cup u_{n+1})$ we must have that $a_{n+1} = 1$ otherwise v was already in $Span(S)$. So therefore we have that:

$$v = \sum_{i=1}^{n} a_i u_i + u_{n+1}$$

We claim that any vector of this form is not already in $Span(S)$. In fact if it were in the $Span(S)$ we would have that:

$$v = \sum_{i=1}^{n} b_i u_i$$

This implies that ($v + v = 0$ because $1 + 1 = 0 + 0 = 0$):

$$v + v = 0 = \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} a_i u_i + u_{n+1} = \sum_{i=1}^{n} (a_i + b_i) u_i + u_{n+1}$$

This would implies that there exists a non trivial linear combination of 0 with distinct vectors of $S \cup u_{n+1}$ since $a_{n+1} = 1$. Thus it contradicts the fact that $S \cup u_{n+1}$ is linearly independent.
Thus every new vectors in $S \cup u_{n+1}$ is of the form:

$$v = \sum_{i=1}^{n} a_i u_i + u_{n+1}$$

Since there are 2^n vectors of the form $\sum_{i=1}^{n} a_i u_i$ since $|Span(S)| = 2^n$ by our induction hypothesis. Therefore, there are 2^n vectors of the form $\sum_{i=1}^{n} a_i u_i + u_{n+1}$. Since these vectors does not belongs to $Span(S)$ we have that:

$$Span(S \cup u_{n+1}) = 2^n + 2^n = 2^{n+1}$$

Hence, we proved that $|Span(S)| = 2^n$ for any linearly independent subset of a vector space V over $\mathbb{Z}_2^k$ such that $|S| = n \geq 1$.
Actually as we proved it before, this is also true for $n = 0$.

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Exercise 4 page 54: *Do the polynomials* $x^3 - 2x^2 + 1{,}4x^2 - x + 3,$ *and* $3x - 2$ *generate* $P_3(\mathbb{R})$*? Justify your answer.*

Let us consider the subset $S = \{x^3 - 2x^2 + 1{,}4x^2 - x + 3, 3x - 2\}$ of the vector space $P_3(R)$. Moreover, we have that $\dim(P_3(\mathbb{R})) = 4$. By the corollary 2 of the replacement theorem, we have that any finite generating set of a vector space of finite dimension n should at least contains n elements. Here we have that $|S| = 3 < \dim(P_3(\mathbb{R})) = 4$. Therefore, S does not generates $P_3(\mathbb{R})$.

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Exercise 5 page 54: *Is* $\{(1,4,-6), (1,5,8), (2,1,1), (0,1,0)\}$ *a linearly independent subset of* $\mathbb{R}^3$*? Justify your answer.*

Let us consider the subset $S = \{(1,4,-6), (1,5,8), (2,1,1), (0,1,0)\}$ of the vector space $\mathbb{R}^3$.

Moreover, we have that $\dim(\mathbb{R}^3) = 3$.

Let n be the dimension of a finite dimensional vector space. By the corollary 2 of replacement theorem, we know that if B is a basis the vector space then $|B| = n$. Moreover a basis generates the vector space by definition. Besides, by the replacement theorem, we know that if L is a linearly independent subset of the vector space that contains m elements and $|G|$ is a generating subset of the vector space, then $|L| \leq |G|$. So in particular for the basis we have $|L| \leq n = |B|$.

In this case, $|S| = 4 > \dim(\mathbb{R}^3) = 3$ so S is not linearly independent.

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Exercise 9 page 55: *The vectors* $u_1 = (1,1,1,1), u_2 = (0,1,1,1), u_3 = (0,0,1,1)$ *and* $u_4 = (0,0,0,1)$ *for a basis for* F^4. *Find the unique representation of an arbitrary vector* (a_1, a_2, a_3, a_4) *in* F^4 *as a linear combination of* u_1, u_2, u_3 *and* u_4.

Since $B = \{u_1, u_2, u_3, u_4\}$ is a basis on F_4 and that $(a_1, a_2, a_3, a_4) \in F^4$, then $\exists!\, b_1, b_2, b_3, b_4$ such that :

$$(a_1, a_2, a_3, a_4) = b_1u_1 + b_2u_2 + b_3u_3 + b_4u_4$$

This implies the following system of equation:

$$\begin{array}{rl} b_1 = a_1 & e_1 \\ b_1 + b_2 = a_2 & e_2 \\ b_1 + b_2 + b_3 = a_3 & e_3 \\ b_1 + b_2 + b_3 + b_4 = a_4 & e_4 \end{array} \quad \rightarrow \quad \begin{array}{rll} b_1 = & a_1 & e_1 \\ b_1 + b_2 = & a_2 & e_2 \\ b_1 + b_2 + b_3 = & a_3 & e_3 \\ b_4 = & a_4 - a_3 & e_4 - e_3 \end{array}$$

$$\begin{array}{rll} b_1 = & a_1 & e_1 \\ b_1 + b_2 = & a_2 & e_2 \\ b_3 = & a_3 - a_2 & e_3 - e_2 \\ b_4 = & a_4 - a_3 & e_4 \end{array} \quad \rightarrow \quad \begin{array}{rl} b_1 = a_1 & e_1 \\ b_2 = a_2 - a_1 & e_2 - e_1 \\ b_3 = a_3 - a_2 & e_3 \\ b_4 = a_4 - a_3 & e_4 \end{array}$$

Therefore, the unique representation of an arbitrary vector $(a_1, a_2, a_3, a_4) \in F^4$ is $(a_1, a_2 - a_1, a_3 - a_2, a_4 - a_3)$ as a linear combination of u_1, u_2, u_3 and u_4.

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Exercise 12 page 55: *Let* u, v *and* w *be distinct vectors of a vector space* V. *Show that if* $\{u, v, w\}$ *is a basis for* V *then* $\{u + v + w, v + w, w\}$ *is also a basis for* V

Let u, v and w be distinct vectors of a vector space V. Suppose that $B = \{u, v, w\}$ is a basis for V.

First notice that $u + v + w, v + w, w$ are distinct vectors since:

- $u + v + w = v + w \Longrightarrow u = 0 \Longrightarrow B = \{0, v, w\} \Longrightarrow B$ is linearly dependent. This contradicts the fact that B is a basis. $\Longrightarrow u + v + w \neq v + w$
- $v + w = w \Longrightarrow v = 0 \Longrightarrow B = \{u, 0, w\} \Longrightarrow B$ is linearly dependent. This contradicts the fact that B is a basis $\Longrightarrow v + w \neq w$
- $u + v + w = w \Longrightarrow u + v = 0 \Longrightarrow$ There exists a non trivial linear combination of 0 with vectors of $B \Longrightarrow B$ is linearly dependent. This contradicts the fact that B is a basis. $\Longrightarrow u + v + w \neq w$

Therefore $u + v + w, v + w, w$ are 3 distinct elements and $B' = \{u + v + w, v + w, w\}$ has 3 elements.

Moreover let $a, b, c \in F$ such that:

$$0 = a(u + v + w) + b(v + w) + cw = au + (a + b)v + (a + b + c)w$$

Since B is a basis we have that the only linear combination of vectors of B that gives the zero vector is the linear combination where all the coefficients equal zero. So by identification we have that

$$\begin{aligned} a &= 0 \\ a + b &= 0 \\ a + b + c &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} a &= 0 \\ b &= 0 \\ c &= 0 \end{aligned}$$

Therefore, the only combination of vectors of B' that gives the zero vector is the combination where all coefficient are equal to 0. Therefore B' is linearly independent.
Since B' is linearly and has exactly 3 elements, B' is a basis of V by corollary 2 of the replacement theorem.

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