proof (of Fubini):

Lemma: If $P = (P_A, P_B)$ is a partition of Q, where P_A is a partition of A & P_B is a partition of B

then $L(f,P) \overset{①}{\le} L(l,P_A) \overset{②}{\le} L(u,P_A) \overset{④}{\le} U(u,P_A) \overset{⑥}{\le} U(f,P)$

$\overset{③}{\le} U(l,P_A) \overset{⑤}{\le}$

Lemma => Thm: f integrable => given $\varepsilon > 0$, can find $P = (P_A, P_B)$ s.t. the two extensions on the left are within ε of each other.

$\Rightarrow U(l,P_A) - L(l,P_A) < \varepsilon \Rightarrow l$ is integrable.

$|\int_Q f - \int_A l| < \varepsilon \Rightarrow \int_Q f = \int_A l$ □.

Proof of lemma ② - ⑥ are trivial:

$u \ge l \Rightarrow$ ②, ⑤. $L \le U \Rightarrow$ ③ ④.

proof of ①: $L(l,P_A) = \sum_{R \in P_A} V(R)\, m_R(l)$

$= \sum_{R\in P_A} V(R) \inf_{x\in R} \underline{\int_{y\in B}} f(x,y) = \sum_{R\in P_A} V(R) \inf_{x\in R} \sup_{P' \text{ of } B} L(f(x,-),P')$

$\ge \sum_{R\in P_A} V(R) \inf_{x\in R} L(f(x,-),P_B) = \sum_{R\in P_A} V(R) \inf_{x\in R}\left(\sum_{R'\in P_B} V(R') \inf_{y\in R'} f(x,y)\right) = \#.$

Aside: f.g. $\inf_x f(x)+g(x) \ge \underbrace{\inf(f(x))}_{a} + \underbrace{\inf(g(x))}_{b}$.

proof: $\inf_x f(x)+g(x) \ge \inf_x a+b = a+b$. □.

$\# \ge \sum_{R\in P_A} V(R) \sum_{R'\in P_B} V(R') \inf_{x\in R} \inf_{y\in R'} f(x,y) = \sum_{R\in P_A}\sum_{R'\in P_B} V(R)V(R') \inf_{(x,y)\in R\times R'} f(x,y)$

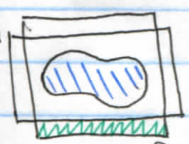
$= \sum_{S\in P} V(S)\, m_S(f). = L(f,P) \Rightarrow$ ①. □.

⑥ is the same. □.

Nov. 21

$\int_Q F \sim \int_S F$

Q_1



Q_2

Def. If S is a bdd subset of $\mathbb{R}^n$, define $\int_S f = \int_Q f 1_S$ where $1_S(x) = \begin{cases} 1 & x\in S \\ 0 & x \notin S \end{cases}$ "the indicator fnctn of S"

where Q is some rectangle containing S.

NTS: Def holds not depend on Q. pf. is almost empty.

Thm 1. f,g integrable over S, $a,b \in \mathbb{R}$, then $\int_S af+bg = a\int_S f + b\int_S g$.

Thm 2. f,g integrable $f \le g \Rightarrow \int_S f \le \int_S g$. □

Thm 3. f integrable $|\int_S f| \le \int_S |f|$

Thm 4. If $f \ge 0$ & integrable, $T \subset S$, then $\int_T f \le \int_S f$

proof of 1: If $a,b \ge 0$, then $m_R(af+bg) \ge m_R(af) + m_R(bg) = a\,m_R(f) + b\,m_R(g)$

$M_R(af+bg) \le M_R(af) + M_R(bg) = a M_R(f) + b M_R(g)$

$\Rightarrow L(af+bg,P) \ge aL(f,P) + bL(g,P)$ $\quad U(af+bg,P) \le aU(f,P) + bL(g,P)$

$\Rightarrow U(af+bg) - L(af+bg,P)$ is small...

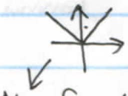
Lemma: $\int(-f) = -\int f$

Proof: $m_R(-f) = -M_R(f) \to L(-f,P) = -U(f,P)$ ... then we can go futher with arbitrary numbers.

proof of 3: $f \le |f| \overset{thm②}{\Rightarrow} \int f \le \int |f|$ $-f \le |f| \Rightarrow -\int f \le \int |f|$. □ (half-credit proof).

Need to show: why is $|f|$ integrable.

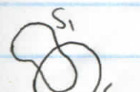
proof: $|f| = \max(f, -f)$.



f, f is cont at $a \Rightarrow |f|$ is cont. at a (because $|f| = Abs \circ f$, Abs & f cont.)

Also $D(|f|) \subset D(f)$. So f integ. $\Rightarrow |f|$ integ. □.

proof of 4: $1_T \le 1_S$. $f 1_T \le f 1_S$, so $\int_T f \cdot \int_Q f \cdot 1_T \le \int_Q f \cdot 1_S = \int_S f$. □



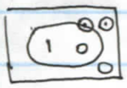
Thm 5. $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$

If $S_1 \cap S_2$ is mea-0, then $\boxed{S_1 \mid S_2}$ $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f$ □.

pf: $1_{S_1 \cup S_2} = 1_{S_1} + 1_{S_2} - 1_{S_1 \cap S_2}$. $\Rightarrow \int_Q f \cdot 1_{S_1 \cup S_2} = \int_Q f \cdot 1_{S_1} + \int_Q f \cdot 1_{S_2} - \int_Q f \cdot 1_{S_1 \cap S_2} = \cdots$

Aside: If S is of mea-0, $\int_S f = 0$.

proof: $\int_S f = \int_Q \underbrace{f \cdot 1_S}_{=0 \text{ except on } S\text{, which is mea } 0} = 0$



Def. If S is bdd & 1_S is integrable, define $V(S) = \int_S 1 = \int_Q 1_S$, and say S is rectifinble.

prop: A bdd S is rect. iff Bd S is mea-0. □.

1. $V(S) \ge 0$. 2. $S_1 \subset S_2$ then $V(S_1) \le V(S_2)$ ...

Nov. 23.

Thm 1 $V(S) \ge 0$. pf. $1_S \ge 0$.

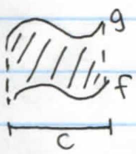
2. $S_1 \subset S_2 \Rightarrow V(S_1) \le V(S_2)$. pf: $1_{S_1} \le 1_{S_2}$.

3. If S_1 & S_2 are rect. then so are $S_1 \cup S_2$ & $S_1 \cap S_2$

$V(S_1 \cup S_2) = V(S_1) + V(S_2) - V(S_1 \cap S_2)$

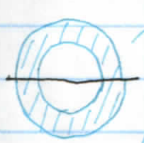
4. If S is rect $V(S) = 0 \Leftrightarrow S$ is of meas-0.

5. If S is rect & $f: S \to \mathbb{R}$ is bdd cont. then $\int_S f$ makes sense.



Thm. If C is a compact rectifinble set in $\mathbb{R}^n$ & $f, g: C \to \mathbb{R}$ are cont. & $f \le g$ on C, then

$D = \{(x,t) : x \in C, f(x) \le t \le g(x)\} \subset \mathbb{R}^{n+1}$ is rectifinable, $V(D) = \int_C g - f$; if $h: D \to \mathbb{R}$ is cont, then $\int_D h = \int_{x \in C} \int_{f(x)}^{g(x)} h(x,t)$



* The only interesting part of the proof is the fact that D is rectifinble

Easy claim: $Bd(D) = \underbrace{\{(x, f(x)) : x \in C\}}_{A} \cup \{(x, g(x)) : x \in C\} \cup \underbrace{\{(x,t) : x \in Bd\, C, f(x) \le t \le g(x)\}}_{B}$

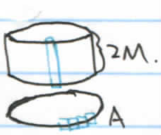
Claim 1: A is mea-0. Claim 2: B is mea-0.

Claim 2: "A cylindar on top of a meas-0 base" is meas-0".

this is kind of B.

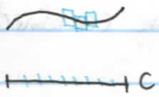
If $A \subset \mathbb{R}^n$ is meas-0 & $f, g: A \to \mathbb{R}$ are bdd (and $f \le g$), then $B = \{(x,t) : x \in A, f(x) \le t \le g(x)\}$ is mea-0

In fact, prove $B_\infty = \{(x,t) : x \in A, t \text{ is anything}\} = A \times \mathbb{R} = \bigcup_n A \times [-n, n]$ is meas-0.



proof: (of claim 2). Assume $|f|, |g| \leq M$, ~~...~~ let $\varepsilon > 0$ be given. Let R_i be rectangles covering A s.t. $\sum v(R_i) < \frac{\varepsilon}{2M}$

then $R_i' = R_i \times [-M, M]$ cover $A \times [-M, M]$ hence cover B, and $\sum v(R_i') = \sum v(R_i) \cdot 2M < \frac{\varepsilon}{2M} 2M = \varepsilon$



Claim 1: "the graph of a cont. function on a compact set is mea-0"

If $C \subset \mathbb{R}^n$ is compact, and $f: C \to \mathbb{R}$ is cont., then $\Gamma_F = \{(x, f(x)) : x \in C\} \subset \mathbb{R}^{n+1}$ is of mea-0.

proof: (Claim 1): let $\varepsilon > 0$, let $\varepsilon_1 = \frac{\varepsilon}{V(Q)}$.

As f is cont. on a compact set, it is also unif. cont. So $\exists \delta > 0$ s.t.

if $x, y \in C$ & $d(x,y) < \delta$, then $|f(x) - f(y)| < \varepsilon_1$

$\Rightarrow$ if R is a rectangle whose diameter is $< \delta$, then $M_R(f) - m_R(f) < \varepsilon_1$.

Find some finite rectangle Q that covers C, and partition it using P s.t. $\operatorname{diam} R < \delta, \forall R \in P$

$\{R \times [m_R(f), M_R(f)] : R \in P\}$ does the job.

Nov. 25

* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$ $\quad \int_0^{\infty} \frac{dx}{x}$ doesn't exist.

Goal: Define $\int_A f$ ($\oint_A f$) where $A \subset \mathbb{R}^n$ is open and possibly not bdd & f is cont. and possibly not bdd.

Def: If $f \geq 0$ set $\oint_A f = \sup\{\oint_D f : D \subset A$ compact & rectifiable$\}$

Otherwise set $f_+(x) = \max(f(x), 0)$ $\quad f_-(x) = \max(-f(x), 0)$ $\quad \rightarrow \begin{cases} f(x) = f_+ - f_- \quad |f| = f_+ + f_- \\ \text{both } f_+ \text{ \& } f_- \text{ are cont.} \end{cases}$

and define $\oint_A f = \oint_A f_+ - \oint_A f_-$ when $\oint_A f_+$ & $\oint_A f_-$ exists.

If all these quantities exists, we will say that " f is E - integrable on A "

* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} := \lim_{n\to\infty} \int_{-n}^{n} \frac{dx}{1+x^2}$

Def. Say that $C_n \nearrow A$ (" C_n rises to A " if A is open) if:

i) $\forall n$, C_n compact & rectifiable. ii) $\forall n$, $C_n \subset \operatorname{int} C_{n+1}$ iii) $\bigcup_{n=1}^{\infty} C_n = A$

e.g. $[-n, n] \nearrow \mathbb{R}$

Thm. If $C_n \nearrow A$ then $\oint_A f$ exists iff $\oint_{C_n} |f|$ is a bdd seq. and in that case $\oint_A f = \lim_{n\to\infty} \oint_{C_n} f$

e.g. $\oint_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{n\to\infty} \int_{-n}^{n} \frac{dx}{1+x^2} = \lim_{n\to\infty} \arctan x \Big|_{-n}^{n} = \lim_{n\to\infty} \operatorname{atg}(n) - \operatorname{atg}(-n) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$

as $[-n, n] \nearrow \mathbb{R} = (-\infty, \infty)$

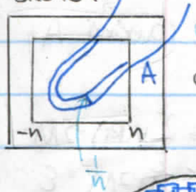
e.g.

$[\frac{1}{n}, n] \nearrow (0, \infty)$ so $\oint_0^{\infty} \frac{dx}{x}$ exist iff $\int_{[\frac{1}{n}, n]} \frac{dx}{x} = \log n - \log \frac{1}{n} = 2\log n$ is bdd, but it's not

So $\oint_0^{\infty} \frac{dx}{x}$ not exists.

Claim: For any A, can find C_n s.t. $C_n \nearrow A$. Sketch (next page).

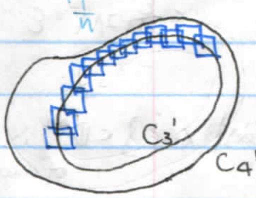
Sketch:



Claim: For any A, can find C_n s.t. $C_n \uparrow A \subset \mathbb{R}^k$

So set $C_n = [-n,n]^k \cap \{x : \overset{\phi\ \text{cont}}{d(x,A^c)} \geq \frac{1}{n}\}$ $\quad$ $\{x : \phi(x) \geq \frac{1}{n}\} = \phi^{-1}([\frac{1}{n},\infty))$

closed & bdd $\quad$ closed. $\quad$ compact $\quad$ closed. $\quad$ closed.



easy $C_n' \subset \text{int } C_{n+1}'$ & $\bigcup C_n' = A$. Then get the C_n's as follows:

← Replace each C_n' by a finite union of rectangles that cover it and are contained in $\text{int } C_{n+1}'$ (using compactness).

proof (thm before) Assume $f \geq 0$, if $\oint_A f$ exists then $\forall n$, $\oint_{C_n} f \leq \oint_A f$ is an inc. seq.

so $\oint_{C_n} f = \int_{C_n} |f|$ has a limit.