TT3. tomorrow 5-7 pm EX300. Mar 13

Claim: If $\phi: \mathbb{R}^n_{x_i} \to \mathbb{R}^n_{y_i}$ & $\omega = f\,dy_{\underline{n}} \in \Omega^{top}(\mathbb{R}^n_{y_i})$, then $\phi^*(\omega) = \det(D\phi)\cdot\phi^* f \cdot dx_{\underline{n}} \in \Omega^{top}(\mathbb{R}^n_{x_i})$

$dy_{\underline{n}} = dy_1 \wedge \cdots \wedge dy_n$ $\quad \phi(x) = \begin{pmatrix}\phi_1(x)\\ \phi_n(x)\end{pmatrix}$

pf: $\phi^*(dy_{\underline{n}}) = \phi^*(dy_1 \wedge \cdots \wedge dy_n) = d(\phi^* y_1)\wedge d(\phi^* y_2)\wedge \cdots \wedge d(\phi^* y_n)$

$= (d\phi_1)\wedge d\phi_2 \wedge \cdots \wedge d\phi_n = \cancel{\left(\frac{\partial\phi_1}{\partial x_1}dx_1 + \frac{\partial \phi_1}{\partial x_2}dx_2 + \cdots\right)}$

$= \left(\frac{\partial\phi_1}{\partial x_1}dx_1 + \frac{\partial\phi_1}{\partial x_2}dx_2 + \cdots + \frac{\partial\phi_1}{\partial x_n}dx_n\right)\wedge\left(\frac{\partial\phi_2}{\partial x_1}dx_1 + \frac{\partial\phi_2}{\partial x_2}dx_2 + \cdots + \frac{\partial\phi_2}{\partial x_n}dx_n\right)$

$\wedge \cdots \wedge \left(\frac{\partial\phi_n}{\partial x_1}dx_1 + \frac{\partial\phi_n}{\partial x_2}dx_2 + \cdots + \frac{\partial\phi_n}{\partial x_n}dx_n\right)$

$dx_5 \wedge dx_5 = 0$

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

$A = (a_{ij})$ is in $M_{n\times n}(\mathbb{R})$

$\det A = \sum_\sigma (-1)^\sigma \prod_{i=1}^n a_{i,\sigma_i}$

$= \sum_{\substack{\text{picking one}\\ \text{term from each line}}}$ (wedge these n terms together) $= \sum_{\substack{\text{Picking 1 term}\\ \text{from each row/col}}}$ (wedge of these terms)

$= \sum_{\sigma\in S_n} \frac{\partial\phi_1}{\partial x_{\sigma_1}}dx_{\sigma_1}\wedge\frac{\partial\phi_2}{\partial x_{\sigma_2}}dx_{\sigma_2}\wedge\cdots = \left(\sum_\sigma(-1)^\sigma\prod_{i=1}^n\frac{\partial\phi_i}{\partial x_{\sigma_i}}\right)dx_{\underline{n}}$

σ_i: the sq# of term in line 1 ...

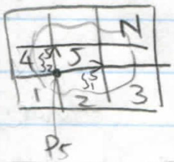
$= \det\left(\frac{\partial\phi_i}{\partial x_j}\right)dx_{\underline{n}} = \det(D\phi)dx_{\underline{n}}$

More generally, if $\omega = \sum_{I\in\binom{n}{k}} a_I\,dy_I \in \Omega^k(\mathbb{R}^n_{y_i})$, then $\phi^*\omega = \sum_{J\in\binom{n}{k}}\sum_{I\in\binom{n}{k}}\left(\phi^* a_I \cdot \det(D\phi)_{J,I}\right)dx_J$

where $(D\phi)_{I,J}$ stands for the $k\times k$ matrix obtained by considering only rows J and col's I of $D\phi$.

$\omega\in\Omega^{top}(\mathbb{R}^k)$ $\quad$ $\mathrm{supp}(\omega) = \overline{\{\text{set of } x\in\mathbb{R}^k \text{ s.t. } \omega(x)\neq 0\}} \subset Q$ a rectangle.

$\omega = f\cdot dx_{\underline{n}}$ $\quad \int_Q\omega \quad \int_{\mathbb{R}^k}\omega := \int_{\mathbb{R}^k} f = \int_Q f$



Intuitive $\fint_Q \omega$ N is big $\sum_{i=1}^N \omega(\xi_1^i, \xi_2^i, \ldots) = \sum_{i=1}^N f(p_i)\cdot(dx_1\wedge dx_2)(\xi_1^i, \xi_2^i) = \sum_{i=1}^n f(p_i)\cdot(\xi_1^i \,|\, \xi_2^i)$

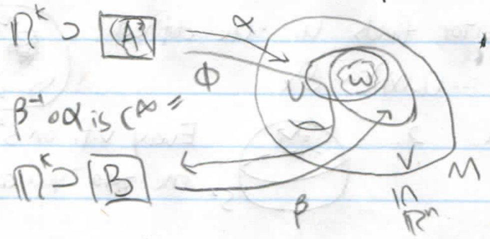
$= \sum_i f(p_i)\cdot \mathrm{vol}(R_i) \underset{\text{big}}{\overset{N \text{ is}}{\approx}} \int_Q f$

Suppose $\omega\in\Omega^k(M)$ and $\mathrm{supp}\,\omega \subset U = \mathrm{im}(\alpha)$ priliminary $\to \fint^{\alpha}_M \omega = \int_{A\subset\mathbb{R}^k}\alpha^*\omega \overset{?}{=} \int_{B\subset\mathbb{R}^k}\beta^*\omega$.

$\alpha = \beta\circ g$. So $\int_A \alpha^*\omega = \int_A(\beta\circ g)^*\omega = \int_A g^*\circ \underbrace{\beta^*\omega}_{top} = \cdots \pm \int_B \beta^*\omega$.

Mar 15

$M^k\subset\mathbb{R}^n$, $\omega\in\Omega^{top}(M)$, $\mathrm{supp}(\omega)\subset U = \mathrm{im}\,\alpha$ $\quad \alpha: A\subset\mathbb{R}^k\to M$ a ~~param~~ patch



$\fint^\alpha_M \omega := \int_A \alpha^*\omega = \int_A f$ where $\alpha^*\omega = f\,dx_{\underline{k}}$

Claim: make sence up to $\pm$!

Precisely, $\fint^\alpha_M\omega = \pm\fint^\beta_M\omega$ our next battle.

provided $\mathrm{supp}\,\omega\subset V = \mathrm{im}\,\beta$, $\beta: B\subset\mathbb{R}^k\to M$ a patch

provided also $U\cap V$ is connected (patch)

pf: $\fint^\beta_M = \int_B g$ where $\beta^* w = g\,dx_{\underline{k}}$

$\fint^\alpha_M\omega = \int_{A' := \alpha^{-1}(U\cap V)}\alpha^*\omega \overset{\alpha = \beta\circ\phi}{=} \int_{A'}(\beta\circ\phi)^*\omega = \int_{A'}\phi^*(\beta^*\omega) = \int_{A'}\phi^*(g\,dx_{\underline{k}}) = \int(g\circ\phi)\det(D\phi)dx_{\underline{k}}$

$= \int_{A'}(g\circ\phi)\cdot\det(D\phi) \underset{\uparrow}{=} \pm\int_{A'}(g\circ\phi)|\det(D\phi)| = \pm\int_{B' = \beta^{-1}(U\cap V)} g = \pm\int_B g = \pm\fint^\beta_M\omega$

on a connected set $\det(D\phi)$ is either unif + or unif −.

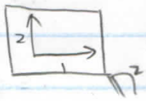
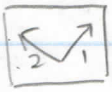
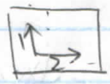
Oriented manifolds. / oriented vector space.

Def. An orientation $\mathcal{O}$ on a f.d ~~V~~ V.S (finite dim. vector space) is a choice of n basis for V, regarded up to positive-det changes of basis.

$(v_1, \dots, v_n)$ is same as $\sim$ $(u_1, \dots, u_n)$ if $\det C_v^u > 0$ where $C_v^u = C_{ij}$ where $u_i = \sum_j c_{ij} v_j$

An oriented v.s is a f.d.v.s along with a choice of ~~ort~~ orientation on it.

Example. 3 a.b $\mathbb{R}^3$, right hdd. left hdd.

2 a.b. $\mathbb{R}^2$ a:  $\mathbb{R}^2$ $\sim$  b: 

1 ab $\mathbb{R}$ 

Claim: Every f.d.v.s. V has exactly 2 distinct orientations

pf: Pick any basis of V: $v_1 \dots v_n$ then $\mathcal{O}_0 = (v_1, \dots, v_n)$ is an orientation of V.

$\mathcal{O}_1 = (-v_1, v_2, \dots, v_n)$ a second orientation of V.

Suppose $u_1 \cdots u_n$ is another basis. if $\det C_{v_1 \cdots v_n}^{u_1 \cdots u_n} > 0$, $u_1 \cdots u_n$ represent the same orientation as $v_1 \cdots v_n$.

otherwises, $\underbrace{C_{-v_1, v_2 \cdots v_n}^{u_1 \cdots u_n}}_{C_1} = \underbrace{C_{v_1 \cdots v_n}^{u_1 \cdots u_n}}_{C_2} \underbrace{\begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{E}$ so $\det(C_1) = \det(C_2)\det(E)$ — $\det(C_1) > 0$

Simily, suppose $\mathcal{O}$ is given by some basis, this orientation is "reversed" (switched to the other)

if 1. You negate any of the vectors in the basis

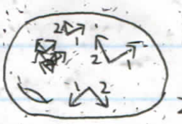
or 2. You swap two vector. $\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

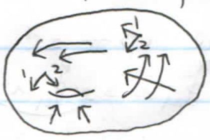
Mar 7

Def: An orientation $\mathcal{O}$ on a f.d.v.s is a choice of an ordered basis for regarded up to a positive-det change basis. $(v_1 \dots v_n) \sim (u_1 \cdots u_n)$ if $\det(C_{v_1 \cdots v_n}^{u_1 \cdots u_n}) > 0$.

Claim: Every f.d.v.s has precisely two orientation. Dmitry: What are they called?

Def: An orientation ~~$\mathcal{O}$~~ $\mathcal{O}$ on a mfd M is a cont. choice of an orientation $\mathcal{O}_x$ on $T_x M$ $\forall x \in M$

Cont: Every $p \in M$ has a nbd W with cont. vector fields $u_1 \cdots u_k$ defined on W s.t. for every $x \in W$, $(u_1(x), u_2(x), \dots u_k(x)) \sim \mathcal{O}_x$  $\subset \mathbb{R}^3$ 2D.

Example 1:  "orientable" choose an orientation "oriented". 2.  S^2 Every v.f. on S^2 has at least one zero "You cannot coimb a coconut"

M^2.  Mobius band. $\leftarrow$ non-orientable. $M^2 \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^4$.

Properties: 1. If $M^k \subset \mathbb{R}^{k+1}$ and $\mathbb{R}^{k+1}$ is orientable, then there is a bijection:

$\{$orientations of $M\}$ $\leftrightarrow$ $\{$cont. varying choice of unit normal vectors to M in $\mathbb{R}^{k+1}\}$

$\forall p \in M$. $n(p) \in T_p\mathbb{R}^{k+1}$ s.t. 1. $n(p) \perp T_pM$ 2. $\|n(p)\| = 1$. 2. $p \mapsto n(p)$ is cont.

Pf: orientation of M: given $p \in M$, have $(u_1 \cdots u_k) \sim \mathcal{O}_p$, find $v \in T_p\mathbb{R}^{k+1}$ s.t.

1: $v \perp u_i$ 2. $\|v\|=1$. 3. $(u_1, \dots u_k, v) \sim$ given orientation of $\mathbb{R}^{k+1}$

Claim: If M^k is orientable, then so is ∂M.

Directive: orient ∂M so that when prependind the outward pointing normal of ∂M to the orientation of ∂M, get the orientation of M.