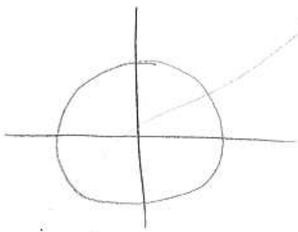
Implicit Function Theorem



Find $y=g(x)$ s.t. $g(a)=b$ &

$\forall x \quad f(x, g(x))=0$

Thm: Given a C^r $f: \mathbb{R}^n_{x_1,\dots,x_n} \times \mathbb{R}^k_{y_1,\dots,y_k} \longrightarrow \mathbb{R}^k$ and

$(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ s.t. $f(a,b)=0$ and

assuming $\frac{\partial f}{\partial y}$ is non-singular

there is a unique C^r

$g: \{\text{nbhd } U \text{ of } a\} \longrightarrow \{\text{nbhd of } b\}$ s.t.

$g(a)=b$ & $\forall z \in U$. $f(z, g(z))=0$

Furthermore $Dg=$

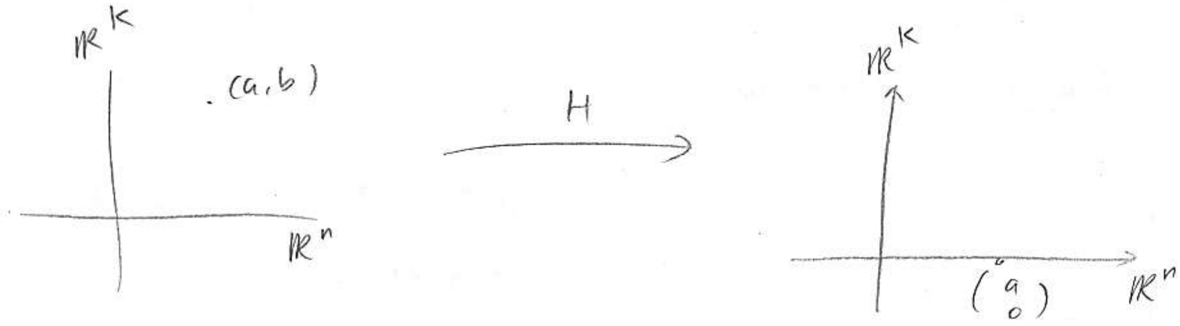
pf: (Brilliant)

$f(z,y)=0 \iff \begin{cases} x=z \\ f(x,y)=0 \end{cases}$ unknown x,y

So now with $H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$

$H: \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$ $\iff H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ 0 \end{pmatrix}$

Solvable using inv. FT.



$$H\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ f(a,b) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad H \text{ is } C^r$$

$$DH\begin{pmatrix} a \\ b \end{pmatrix}$$

$$DH = \begin{pmatrix} \frac{\partial H_1}{\partial x} & \frac{\partial H_1}{\partial y} \\ \frac{\partial H_2}{\partial x} & \frac{\partial H_2}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\frac{\partial f}{\partial y} = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_k} \end{pmatrix}}_{k} \left.\vphantom{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}\right\} k$$

$$\frac{\partial f}{\partial x} = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}}_{n} \left.\vphantom{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}\right\} k$$

$$\frac{\partial x}{\partial x} = \underbrace{\begin{pmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_n} \end{pmatrix}}_{n} \Bigg\} n$$

$$= \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$= I_{nn}$$

$$\frac{\partial x}{\partial y} = 0$$

$$DH = \begin{pmatrix} I & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

Invertible? $\iff \frac{\partial f}{\partial y}$

$(\Rightarrow)$ $\exists H^{-1}$ in some nbhd of (a,b)

set $g(z) = \pi_2\left(H^{-1}\begin{pmatrix} z \\ 0 \end{pmatrix}\right)$. $\quad \pi_2 : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$

$(x,y) \longmapsto y$

easy to verify $g(a)=b$ & $\forall_x\ f(x,g(x)) = 0$. g is C^r.

$0 = f(x, g(x))$

$\downarrow$ taking D

$0 = D[\qquad]$

$$\mathbb{R}^n_x \xrightarrow[\begin{pmatrix} \frac{\partial x}{\partial x} \\ \frac{\partial g}{\partial x} \end{pmatrix}]{x \mapsto \binom{x}{g(x)}} \begin{matrix} \mathbb{R}^n \\ \times \\ \mathbb{R}^k \end{matrix} \xrightarrow[\left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right)]{\binom{x}{y} \to f(x,y)} \mathbb{R}_k$$

$$= \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} I \\ Dg \end{pmatrix}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} Dg$$

$$\Rightarrow Dg = \underbrace{\underbrace{\left(\frac{\partial f}{\partial y}\right)^{-1}(x, g(x))}_{k \times k} \underbrace{\left(\frac{\partial f}{\partial x}\right)(x, g(x))}_{k \times n}}_{k \times n}$$