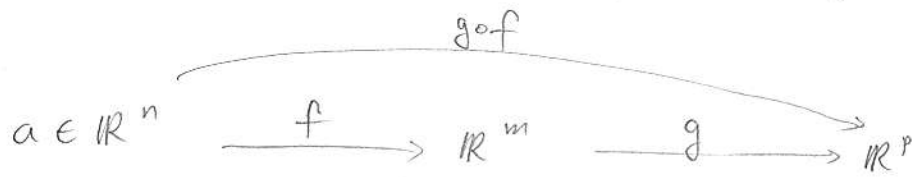


$$f(a+h) = f(a) + Df(a)h + \varphi(h), \quad \varphi \in o(h)$$



Assumptions $a \in \mathbb{R}^n$

f is diff. at a

g is diff. at $f(a)$

Thm:

$$D(g \circ f)(a) = \underbrace{Dg(f(a))}_{p \times m} \cdot \underbrace{Df(a)}_{m \times n}$$

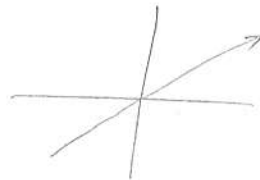
$p \times n$

e.g:

$$\mathbb{R}_+ \xrightarrow{f} \mathbb{R}_{x,y}^2 \xrightarrow{g} \mathbb{R}$$

$t \mapsto (t, t) \quad (x, y) \mapsto x^y$

$$f(t) = (t, t)$$



$$(g \circ f)(t) = g(f(t)) = g(t, t) = t^t$$

$$D(g \circ f)(t) = ((t^t)')$$

$$= BA$$

$$= (t t^{-1} t^t \log t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= t^t (1 + \log t)$$

$$Df(t) = \begin{pmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A$$

$$Dg \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

$$= (\gamma x^{\gamma-1} \quad x^{\gamma} \log x)$$

$$Dg(f(t)) = (t t^{t-1} \quad t^t \log t) = B$$

pf: $b = f(a)$

$$f(a+h) = f(a) + Df(a) \cdot h + \phi(h) \quad \phi \in o(h)$$

$$g(b+k) = g(b) + Dg(b) \cdot k + \gamma(k) \quad \gamma \in o(k)$$

$$(g \circ f)(a+h) = g(f(a+h))$$

$$= g \left(\underbrace{f(a)}_b + \underbrace{Df(a) \cdot h + \phi(h)}_k \right)$$

$$= g(b) + Dg(b) \cdot (Df(a) \cdot h + \phi(h)) + \gamma(Df(a) \cdot h + \phi(h))$$

$$= (g \circ f)(a) + \underbrace{Dg(f(a)) \cdot Df(a) \cdot h}_{\times |h|} + \underbrace{Dg(b) \cdot \phi(h) + \gamma(Df(a) \cdot h + \phi(h))}_{\times |h|}$$

Claim: $\gamma(k) \in o(k)$, namely $\lim_{k \rightarrow 0} \frac{\gamma(k)}{|k|} = 0$

$$\text{pf: } \frac{|Dg(b) \cdot \phi(h)|}{|h|} \leq m |Dg(b)| \frac{|\phi(h)|}{|h|} \rightarrow 0 \quad \text{as } h \rightarrow 0$$

$|A v|_{\text{sup}} \leq n \cdot |A|_{\text{sup}} \cdot |v|_{\text{sup}}$
 $A \in M_{n \times n}$

Cor: Composition of two C^r fcts. is C^r

$$f, g \in C^r \Rightarrow g \circ f \in C^r$$

pf: By induction on r .