$$f(a+h) = f(a) + Df(a)h + \varphi(h) \qquad \varphi \in o(h)$$

$$\begin{tikzcd} a \in \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^p \end{tikzcd}$$

Assumptions $a \in \mathbb{R}^n$

f is diff. at a

g is diff. at $f(a)$

Thm: $$\underbrace{D(g\circ f)(a)}_{p\times n} = \underbrace{\underbrace{Dg(f(a))}_{p\times m} \cdot \underbrace{Df(a)}_{m\times n}}_{p\times n}$$

e.g.: $$\mathbb{R}_t \xrightarrow[t \longmapsto (t,t)]{f} \mathbb{R}^2_{x,y} \xrightarrow[(x,y) \longmapsto x^y]{g} \mathbb{R}$$

$f(t) = (t,t)$

$(g\circ f)(t) = g(f(t)) = g(t,t) = t^t$

$D(g\circ f)(t) = ((t^t)')$

$= BA$

$= (t\,t^{-1} \;\; t^t \log t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= t^t(1 + \log t)$

$$Df(t) = \begin{pmatrix} \frac{\partial t}{\partial t} \\ \frac{\partial t}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A$$

$$Dg\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} y x^{y-1} & x^y \log x \end{pmatrix}$$

$$Dg(f(t)) = \begin{pmatrix} t t^{t-1} & t^t \log t \end{pmatrix} = B$$

pf: $b = f(a)$

$$f(a+h) = f(a) + Df(a)\cdot h + \phi(h) \qquad \phi \in o(h)$$

$$g(b+k) = g(b) + Dg(b)\cdot k + \gamma(k) \qquad \gamma \in o(k)$$

$$(g \circ f)(a+h) = g(f(a+h))$$

$$= g(\underbrace{f(a)}_{b} + \underbrace{Df(a)\cdot h + \phi(h)}_{k})$$

$$= g(b) + Dg(b)\cdot(Df(a)\cdot h + \phi(h)) + \gamma(Df(a)\cdot h + \phi(h))$$

$$= (g\circ f)(a) + \underline{Dg(f(a))\cdot Df(a)\cdot h} + \underbrace{Dg(b)\cdot\phi(h) + \gamma(Df(a)\cdot h + \phi(h))}_{\lambda(h)}$$

Claim: $\lambda(\alpha) \in o(h)$. namely $\lim_{h\to 0} \frac{\lambda(h)}{|h|} = 0$

pf: $\frac{|Dg(b)\cdot\phi(h)|}{|h|} \le m\,|Dg(b)|\,\frac{|\phi(h)|}{|h|}$

$\longrightarrow 0 \quad$ as $\quad h \to 0$

$$\boxed{\begin{array}{l} |Av|_{sup} \le n\cdot|A|_{sup}\cdot|v|_{sup} \\ A \in M_{m\times n} \end{array}}$$

Cor: Composition of two C^r fcts. is C^r

$$f, g \in C^r \Rightarrow g \circ f \in C^r$$

pf: By induction on r.