Oct 2, 2012 Mat267 

## Chain Law

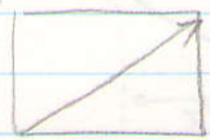
$$\begin{tikzcd} \mathbb{R}_t \arrow[r, "\gamma"] \arrow[rr, bend right, "g = f\circ\gamma"'] & \mathbb{R}^n_{x_1,\dots,x_n} \arrow[r, "f"] & \mathbb{R} \end{tikzcd} \qquad \gamma = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$g(t+\varepsilon) = f(\gamma(t+\varepsilon))$$
$$= f(\gamma_1(t+\varepsilon), \dots, \gamma_n(t+\varepsilon))$$
$$= \underbrace{f_1 \circ \gamma_1 \cdot \varepsilon}_{\text{change due to change in 1st coor.}} + f_2\gamma_2 \cdot \varepsilon + \cdots$$

$$= \left( \sum \frac{\partial f}{\partial x_i} \cdot \gamma_i'(t) \right) \varepsilon$$

$$\begin{tikzcd} \mathbb{R}_t \arrow[r, "(t,t)", "\gamma"'] \arrow[rr, bend right, "g"'] & \mathbb{R}^2_{x,y} \arrow[r, "f", "x^y"'] & \mathbb{R} \end{tikzcd} \qquad \gamma(t) = \begin{pmatrix} t \\ t \end{pmatrix},\ f(x,y) = x^y$$

$$g = f \circ \gamma = t^t \qquad \gamma_1 = t,\ \gamma_2 = t,\ \gamma_1' = 1\ \ \gamma_2' = 1$$



$$g^{-1}(t) = \frac{\partial f}{\partial x} \cdot 1 + \frac{\partial f}{\partial y} \cdot 1$$
$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

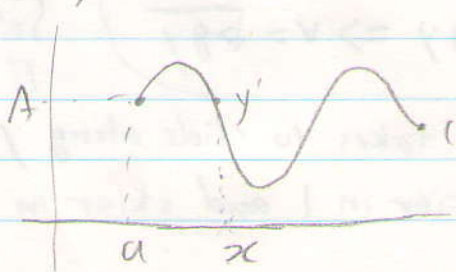
### Problem

Minimize

$$J(y) = \int_a^b F(x, y, y')\,dx$$

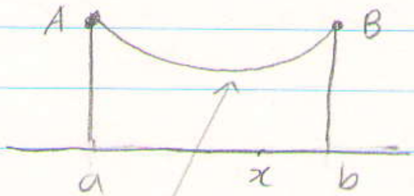
$F: \mathbb{R}^3 \to \mathbb{R}$ sufficiently differentiable

Among all sufficiently differentiable functions $y: [a,b] \to \mathbb{R}$ s.t. $y(a) = A$ and $y(b) = B$



examples

① The "power line" problem

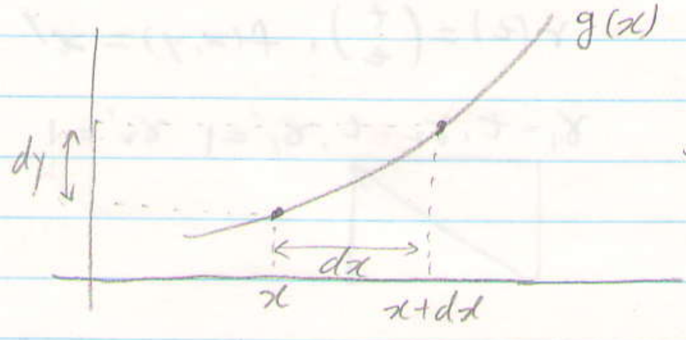



$J(y)$ = potential energy of $y$

The potential energy of a tiny piece of string is proportional to its height and its mass

$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$J(y) = \int_a^b y \left( \begin{array}{c} \text{mass of string} \\ \text{between} \\ x \text{ and } x+dx \end{array} \right) \propto \int_a^b y \left( \begin{array}{c} \text{length of} \\ y \text{ between} \\ x \text{ and } x+dx \end{array} \right) = \int_a^b y \cdot \sqrt{1+(y')^2}\, dx$$




$$\text{length} = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\rightsquigarrow \sqrt{1+(y')^2}\, dx$$

$$F(x_1, x_2, x_3) = x_2 \cdot \sqrt{1+x_3^2}$$

② $\mathcal{L}(q) = \int \frac{1}{2} m \dot{q}^2 - V(q(t))$

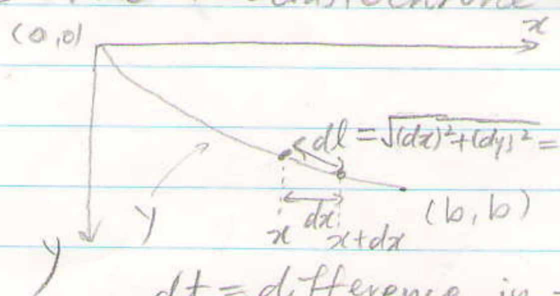
"J" "y" "x"

kinetic energy

$V: \mathbb{R} \to \mathbb{R}$ potential energy at $q$

$q$ : position of particle as function of time.

③ The Brachistochrone




$$dl = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+y'^2}\,dx$$

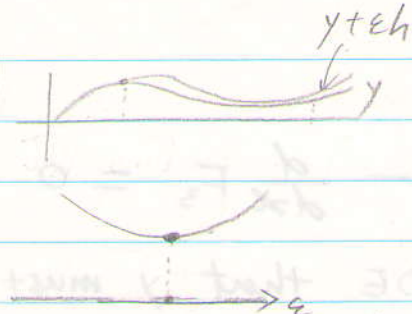
$$\frac{1}{2} m v^2 = mgy \Rightarrow v = \sqrt{2gy}$$

$$\int_0^b \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx = \int_0^b dt = T(y)$$

$T(y)$ = time it takes to slide along $y$

$dt$ = difference in time between skier in 1 and skier in 2

$$= \frac{dl}{v} = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx$$



J(y+εh) as a function of ε has min. at ε=0

## Euler - Lagrange Equation

$y$ is an extremum

$\Rightarrow \forall h,\ h:[a,b]\to\mathbb{R},\ h(a)=0,\ h(b)=0$

$$\frac{d}{d\varepsilon}J(y+\varepsilon h)\Big|_{\varepsilon=0}=0$$

$$\frac{d}{d\varepsilon}J(y+\varepsilon h)\Big|_{\varepsilon=0}=\frac{d}{d\varepsilon}\int_a^b F(x,y+\varepsilon h,y'+\varepsilon h')\,dx\Big|_{\varepsilon=0}$$

$$=\int_a^b \frac{d}{d\varepsilon}F(x,y+\varepsilon h,y'+\varepsilon h')\,dx\Big|_{\varepsilon=0}$$

$$=\int_a^b (F_1\cdot 0+\underbrace{F_2}_{F_2(x,y+\varepsilon h,y'+\varepsilon h')}\cdot h+\underbrace{F_3}_{=F_3(\dots)}\cdot h')\,dx\Big|_{\varepsilon_0}$$

$$=\int_a^b (F_2(x,y+y')h+F_3(\cdots)h')\,dx$$

integrate by parts

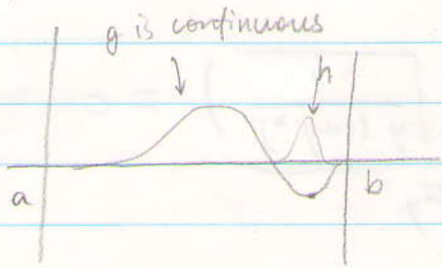
$$\cong\int_a^b \left(F_2h-\left(\frac{d}{dx}F_3\right)h\right)dx+\cancel{\text{boundary terms } F_3\cdot h\Big|_a^b}\ \ \begin{matrix}h(a)=0\\h(b)=0\end{matrix}=0$$

$$=\int_a^b \underbrace{\left(F_2-\frac{d}{dx}F_3\right)}_{g(x)}\cdot h(x)\,dx \quad (\text{should be } 0\ \forall h)$$

$\Rightarrow \forall h,\ \int_a^b g\cdot h=0$

$\Rightarrow g\equiv 0$

proof



Then,

$$F_2 - \frac{d}{dx} F_3 = 0 \quad \text{(\underline{Euler-Lagrange Equation})}$$

$\Rightarrow$ ODE that $y$ must satisfy

same equation

<u>example</u>

$$J = \int_a^b \underbrace{\left[\frac{1}{2} m (y')^2 - V(y)\right]}_{F(x,y,y')} dx$$

$$F(u_1, u_2, u_3) = \frac{1}{2} m u_3^2 - V u_2$$

$$F_2 = -V'(u_2) \quad \text{and} \quad F_3 = m u_3$$

<u>E-L</u>: $F_2 - \frac{d}{dx} F_3 = 0$

$$\Rightarrow -V'(y) - \frac{d}{dx}(my') = 0$$

$$my'' = -V'(y)$$

$$ma = F$$

<u>Note</u>

$$F_y - \frac{d}{dx} F_{y'} = 0 \quad \text{(\underline{Euler-Lagrange Equation})}$$

<u>example (Brachistochrone)</u>

$$F = \sqrt{\frac{1+y'^2}{y}}$$

<u>E-L</u>: $\underbrace{-\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}}}_{F_y} - \frac{d}{dx}\underbrace{\left(2y' \cdot \frac{1}{2} \cdot \sqrt{\frac{1}{y(1+y'^2)}}\right)}_{F_{y'}} = 0$