Def: $B(x, \varepsilon) = \{ y \mid d(x, y) < \varepsilon \}$

Def: U open $\iff \forall x \in U \ \exists \varepsilon > 0 \quad B(x, \varepsilon) \subset U$.

Def: U closed $\iff U^c = X \setminus U$ open

Thm 1a: 1. $\emptyset, X$ open

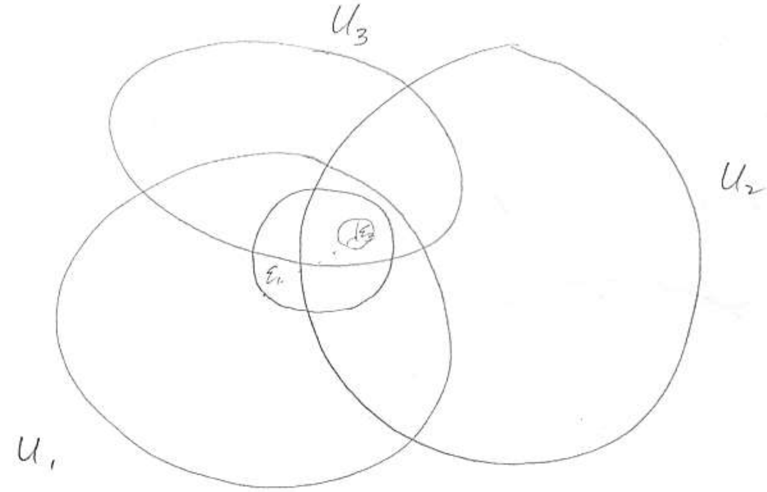
2. $\forall \alpha \in I$, U_α open $\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ open

3. $\forall 1 \leq i \leq n$, U_i open $\Rightarrow \bigcap_{i=1}^{n} U_i$ open

pf: 1. done.

2. Let x be some pt in $\bigcup_{\alpha \in I} U_\alpha$, meaning $\exists \alpha_0 \in I$ s.t. $x \in U_{\alpha_0}$. Then let $\varepsilon > 0$ be s.t. $B(x, \varepsilon) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in I} U_\alpha$ □

3. Let $x \in \bigcap_{i=1}^{n} U_i \Rightarrow \forall_i$, $x \in U_i$ for each i choose ε_i



$B(x, \varepsilon_i) \subset U_i$

Let $\varepsilon = \min_{1 \leq i \leq n} \varepsilon_i > 0$ & $\forall_i \ B(x, \varepsilon) \subset B(x, \varepsilon_i) \subset U_i \Rightarrow B(x, \varepsilon) \subset \bigcap U_i$

Thm : (DeMorgan)

$$\left(\bigcup_{\alpha \in I} A_\alpha \right)^c = \bigcap_{\alpha \in I} A_\alpha^c$$

$$\left(\bigcap_{\alpha \in I} A_\alpha \right)^c = \bigcup_{\alpha \in I} A_\alpha^c$$

Thm 1b: 1. ϕ, X closed

2. $\forall \alpha$. F_α closed $\Rightarrow$ $\bigcap_\alpha F_\alpha$ closed

3. $\forall 1 \le i \le n$. F_i closed $\Rightarrow$ $\bigcup_{i=1}^n F_i$ closed.

Thm 4a.b: $\mathbb{R}^n$. $d_1(x,y) = \|x-y\|$. $d_2(x,y) = |x-y|$

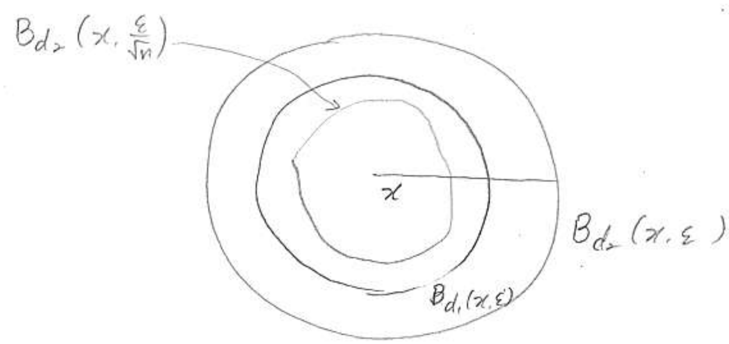
a. U open rel. d_1 $\Leftrightarrow$ U open rel. d_2

b. F closed rel d_1 $\Leftrightarrow$ F closed rel. d_2.

pf: a. Assume U is d_1-open.

Let $x \in U \Rightarrow \exists \varepsilon > 0$ s.t. $B_{d_1}(x, \varepsilon) \subset U$.

$$d_2 \le d_1 \le \sqrt{n}\, d_2$$



claim: $B_{d_2}(x, \varepsilon) \supset B_{d_1}(x, \varepsilon) \supset B_{d_2}(x, \frac{\varepsilon}{\sqrt{n}})$

pf: $y \in B_{d_1}(x, \varepsilon)$. Then,

$d_2(x, y) \le d_1(x, y) < \varepsilon$

So $y \in B_{d_2}(x, \varepsilon)$

if $y \in B_{d_2}(x, \frac{\varepsilon}{\sqrt{n}})$. then $d_2(x, y) < \frac{\varepsilon}{\sqrt{n}}$

$\sqrt{n}\, d_2(x, y) < \varepsilon$

$d_1(x, y) \le \sqrt{n}\, d_2(x, y) < \varepsilon$

$\Rightarrow y \in B_{d_1}(x, \varepsilon)$

Now $B_{d_2}(x, \frac{\varepsilon}{\sqrt{n}}) \subset B_{d_1}(x, \varepsilon) \subset U$

$\Rightarrow U$ is d_2-open.

Need to show: d_2-open $\Rightarrow$ d_1-open

d_1-closed $\Leftrightarrow$ d_2-closed.

□

Thm 2: X metric sp. $Y \subset X$ is also a metric sp.

1. $U \subset Y$ open iff $\exists V \subset X$ s.t. V open in X & $U = V \cap Y$
2. $F \subset Y$ closed iff $\exists G \subset X$ s.t. G closed in X & $G \cap Y = F$

□

e.g: $X = \mathbb{R}^2$ $Y = \mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$.