Recall

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad C_n = \sum_{k=1}^{n} C_{n-k}\cdot C_{k-1}$$

$$F = F_c = \sum C_k x^k = \frac{1-\sqrt{1-4x}}{2x}$$

$$(1+y)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} y^k$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$\underset{\substack{\uparrow \\ (1+y)^{\frac{1}{2}}}}{\binom{\frac{1}{2}}{k}} = \cdots = \frac{(-1)^{k-1}}{2^k\cdot k!}\underbrace{(1\cdot 3\cdot 5\cdots(2k-3))}_{(2k-3)!!} \overset{(\star)}{=} ?$$

Aside: Understand $(2n-1)!!$

Sol ①: $(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1)$

$$= \frac{(2n)!}{2\cdot 4\cdot 6\cdot 8\cdots(2n)}$$

$$= \frac{(2n)!}{2^n(1\cdot 2\cdots n)}$$

$$= \frac{(2n)!}{2^n\cdot n!}$$

Sol ②: Start by asking a question.

How many ways to pair $2n$ people?

$$\underbrace{1 \quad 2}\quad 3 \quad 4 \quad 5 \quad 6$$

Ans 1: $(2n-1)(2n-3)(2n-5)(2n-7)\ldots$

Ans 2: $\dfrac{(2n)!}{2^n \cdot n!} \Rightarrow (2n-1)!! = \dfrac{(2n)!}{2^n \cdot n!}$

Now $(2k-3)!! = \dfrac{2(k-1)!!}{2^{k-1}(k-1)!} \overset{(※)}{\Longrightarrow} ? = \dfrac{(-1)2^{k-1}(2k-2)!}{2^{2k-1}\cdot k((k-1)!)^2} = \dfrac{(-1)^{k-1}}{2^{2k-1}\cdot k}\dbinom{2k-2}{k-1}$

with $\dbinom{\frac{1}{2}}{0} = 1$

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}}$$

$$= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{\alpha} (-4)^k x^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}(k)} \binom{2k-2}{k-1} (-4)^k \cdot x^k$$

$$= 1 - \sum_{k=1}^{\infty} \frac{2}{k} \binom{2k-2}{k-1} \cdot x^k$$

$$F = \frac{1-\sqrt{1-4x}}{2x}$$

$$= \frac{\sum_{k=1}^{\infty} \frac{2}{k}\binom{2k-2}{k-1} x^k}{2x}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} x^{k-1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n$$

$$= \sum_{n \geq 0} C_n \cdot x^n$$

$= \frac{1}{n+1}\binom{2n}{n}$

Q: How many way to the top of an n-stair staircases if in each leap, you can climb one or two steps? Call answer $F_n$

A: $F_0 = 1$

$F_1 = 1$

$F_2 = 2$

$F_3 = 3$, 5, 8, ...

$F_n = F_{n-1} + F_{n-2}$