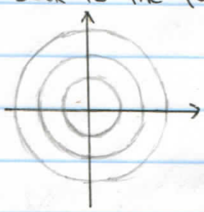
Remark: we use 2-var. integral to find $(x^x)'$

Back to the function $e^{-(x^2+y^2)/2}$. it is constant at circles. (as $(x^2+y^2) = |(x,y)|^2$. distance from $(0,0)$)



Partition of annualar. $G_2 \approx \sum_i V(A_i) f(x_i) \quad x_i \in A_i$

Suppose $= \sum_i V(\text{circle of radius } r_i)\, e^{-r_i^2/2} \cdot r_i$ ← length of circle

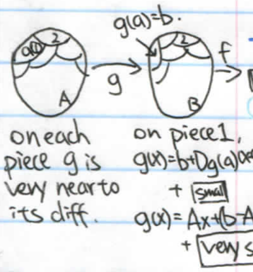
$= \int_r e^{-r^2/2}\, 2\pi r$

$= 2\pi$

Define $G_N = \int_{\mathbb{R}^n} e^{-x^2/2} = (G_1)^n = (2\pi)^{n/2}$

$= \int_0^\infty dr\, e^{-r^2/2}\, r^{n-1}\, \mathrm{vol}(S_p^{\,n-1})$ ← use the riddle to solve this.

Dec. 5.




**Thm:** (C of V. Sil) $\int_B f = \int_A (f\circ g)\,|\det Dg|$ (and each exists iff the other exists). $J_g(x) = \det Dg(x)$.

Geometry Part. True if $g$ is affine linear. $g(x) = Ax + b$.

Analysis. Therefore thm is true for any $g$.

$X \xrightarrow{E^1} \xrightarrow{E^2} \xrightarrow{E^3} \cdots \xrightarrow{A} Ax \xrightarrow{+b} Ax+b$. $E'(i,j;\lambda)$ $D(j;\lambda)$ $P(i,j)$

Claim: $\underset{\text{open}}{A} \ni x \xrightarrow[\text{diffeo}]{g_1} \underset{\text{open}}{B} \ni y \xrightarrow[\text{diffeo}]{g_2} \underset{\text{open} \subseteq \mathbb{R}^n}{C} \ni z \xrightarrow{f} \mathbb{R}$. $\quad h: B \to \mathbb{R}$

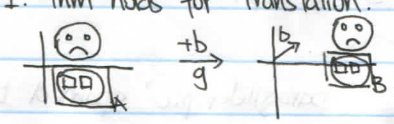
If thm is true for $g_1$ & for $g_2$, then it is true for $g_2 \circ g_1$

Pf. $\int_{z\in C} f(z) \overset{\text{by thm for } g_2}{=} \int_{y\in B} \underbrace{f(g_2(y)) \cdot |\det(Dg_2(y))|}_{\text{function of } y} \overset{\text{by thm for } g_1}{=} \int_{x\in A} f(g_2(g_1(x))) \cdot |\det Dg_2(g_1(x))| \cdot |\det(Dg_1(x))|$

$= \int_{x\in A} f(g_2\circ g_1)(x)) \cdot |\det(\underbrace{Dg_2(g_1(x)) \cdot Dg_1(x)}_{D(g_2\circ g_1)(x)})|$

$= \int_A f\circ(g_2\circ g_1) \cdot (J_{g_2\circ g_1})$

1. thm holds for translation. $J_g = 1$




$\int_{x\in A} f(x+b)\, 1 = \int_{y\in B} f(y) \qquad B = A+b$.

pf. (In short) There is a bijection between partitions of A & partitions of B, $P \to P+b$.

$(f\circ g, P) \mapsto (f, P+b)$.

2. $g(x) = Ex$. $E = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$ $Dg = E$. $|J_g| = |\det E| = 1$.

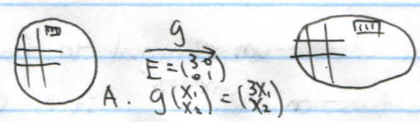
$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $E\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$. $E\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_j \\ x_i \\ x_n \end{pmatrix}$.

Notes. $\int_{x\in A} f(Ex) \cdot 1 = \int_{y\in EA} f(y)$. Any partition $P$ of $A$ becomes a partition $EP$ of $B = EA$

$R \in P \rightsquigarrow ER \in EP$. $\mathrm{Vol}(R) = \mathrm{Vol}(ER)$ $\quad L(f\circ E, P) = L(f, EP)$. ...

3. $g(x) = Ex$. $E = \begin{pmatrix} 1 & & \\ & c & \\ & & 1 \end{pmatrix}$ $c \neq 0$. $Dg = E$. $|J_g| = |\det E| = c$.

NTS. $\int_{x\in A} f(Ex)\cdot|c| = \int_{y\in EA} f(y)$



$\xrightarrow[E=\begin{pmatrix}3&0\\0&1\end{pmatrix}]{g}$ A. $g\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}3x_1\\x_2\end{pmatrix}$ B=EA Vol(ER) = 3vol(R).

Dec. 7

**Thm.** Given $(A) \xrightarrow{g} (B) \xrightarrow{f} \mathbb{R}$, $\int_B f = \int_A (f\circ g)|\det D_g|$

Lemma: True for $g_1, g_2 \Rightarrow$ true for $g_1\circ g_2$.

Lemma: True for affine linear maps: $g(x) = Ax+b$. $b\in\mathbb{R}^n$, $A\in M_{n\times n}$ invertible.

Done for: ① $g(x)=x+b$. ② coord. sums. $|J|=1$. ③ coord rescales. $\begin{pmatrix}1&&\\&c&\\&&1\end{pmatrix}$ $|J|=|c|$.

Remaining cases: WLOG $A = E_{i,j,c} = \begin{pmatrix}1&&c\\&1&\\&&1\end{pmatrix}$ $E_c = \left(\begin{smallmatrix}1&c\\0&1\end{smallmatrix}\Big|\,I\right)$.

$(A) \xrightarrow[g(x)=E_i x]{} (B) \xrightarrow{f} \mathbb{R}$ $J = \left|\det\begin{pmatrix}1&c&\\&1&\\&&1\end{pmatrix}\right| = 1$. $g\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}x_1+cx_2\\x_2\\ \vdots\\ x_n\end{pmatrix}$

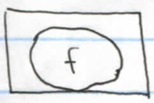
$\int_{y\in B} f(y) \overset{?}{=} \int_{x\in A}(f\circ g)(x)\cdot 1$

LHS $= \int_{y\in B} f(y_1,\dots,y_n)$ RHS $= \int_{x\in A} f(x_1+cx_2, x_2, x_3,\dots,x_n)$.

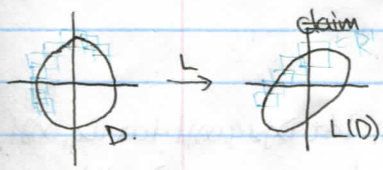
NTS: 1. (L integrable) $\Leftrightarrow$ (R integrable) 2. Assuming integ, L=R.

proof of 2: write $x=(x_1,x')$ $x'=(x_2,\dots,x_n)$.

$R \overset{\text{Fubini}}{=} \int_{x'}\left(\int_{x_1} f(x_1+cx_2,x')\right) \overset{①}{=} \int_{x'}\left(\int_{x_1} f(x_1,x')\right) \overset{\text{Fubini}}{=} \int_x f = L$



Now proof of 1: $D(f\circ E_c) = \{x : E_c(x)\in D(f)\} = E_c^{-1}(D(f)) = E_{-c}(D(f))$.



~~claim~~ claim: If $L:\mathbb{R}^n\to\mathbb{R}$ is linear & $D\subseteq\mathbb{R}^n$ is meas-0, then $L(D)$ is meas-0.

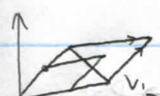
NTS: $\exists K\in\mathbb{R}$ s.t. for any rectangle R, can find a rectangle R' s.t. $L(R)\subset R'$ & $\mathrm{vol}(R')\le K\,\mathrm{vol}(R)$.

WLOG, R is a box: a rectangle whose sides are all the same.

WLOG. $R=[0,1]^n$ $L([0,1]^n)\subset[-M,M]^n$ w/ $M = n\cdot\|A\|_{sup}$.

We know, $\|Av\|_{sup} \le n\cdot\|A\|_{sup}\|v\|_\infty$ if $v\in[0,1]^n$, $\|Av\|_{sup}\le n\cdot\|A\|_{sup}$.

$\mathrm{vol}(L[0,1]^n) \le (2n\|A\|_{sup})^n$ $K = (2n\|A\|_{sup})^n$



*. $v_1\dots v_n\in\mathbb{R}^n$, $P(v_1,\dots,v_n) := \{\sum a_i v_i ;\ 0\le a_i\le 1\ \forall i\}$ "paragllelepipic" spanned by $v_1\dots v_n$"

Claim: $\mathrm{vol}(P(v_1,\dots,v_n)) = |\det(v_1|\cdots|v_n)|$ Geometric meaning to det!

proof: $P = A\cdot[0,1]^n$ $(a_1,\dots,a_n)\in[0,1]^n$ where $A=(v_1|\cdots|v_n)$.

$\mathrm{Vol}(P) = \int_{\mathbb{R}^n} 1_P = \int_{\mathbb{R}^n} 1_{[0,1]^n}\circ A^{-1} = |\det(A)|\int(1_{[0,1]^n}\circ A^{-1})|\det(A^{-1})|$

$= |\det(A)|\int 1_{[0,1]^n} = |\det A|\ \mathrm{vol}\,[0,1]^n = |\det A|\cdot 1.$