$C_n$ : In how many soccer histories ending $(n,n)$ team A is never behind?

"A - dominant"

B

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | 0 | 42 | |
| | | | 0 | 14 | 42 | |
| 0 | 0 | 0 | 5 | 14 | 28 | |
| 0 | 0 | 2 | 5 | 9 | 14 | 20 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |

O A

· recurrsion formula

Formula for $a_n$ given $a_0, \cdots, a_{n-1}$

Initial condition: value of $a_0$ or $a_0$ and $a_1$

e.g: 0. $a_n = a_{n-1} + 1$ : $a_0 = 0$ $a_1 = 1$ . $a_2 = 2$

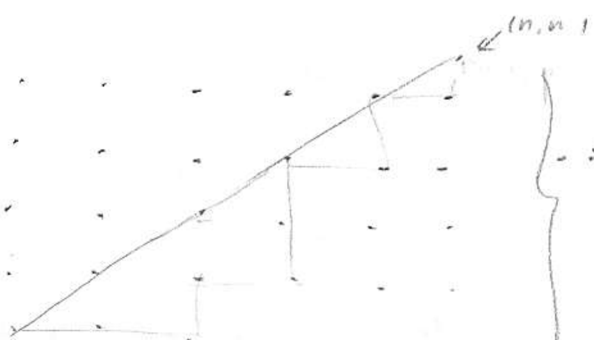
1. $a_n = a_{n-1} + n$ : $a_0 = 0$ . $a_1 = 1$ . $a_2 = 3$ . $a_3 = 6$

2. $a_n = n \cdot a_{n-1}$ : $a_0 = 1$ $\longrightarrow a_n = n!$

Fibonacci $\to$ 3. $F_0 = F_1 = 1$ , $F_n = F_{n-1} + F_{n-2}$ : $F_n = 1, 1, 2, 3, 5, 8, \ldots$

On an A-dominant path from $\underbrace{(0,0) \text{ to } (n,n)}_{a_0 = 1 \text{ (For } n>0)}$ ,

Consider the first tie $(k,k)$ after $(0,0)$




· # of possibilities after the first tie assume by it is $(k,k)$?

$\Rightarrow C_{n-k}$ : ordinary game.

· # of possibilities before the first tie assume by it is $(k,k)$:

$\Rightarrow C_{k-1}$ : not ordinary

So $C_n = \sum_{k=1}^{n} C_{n-k} \cdot C_{k-1}$ with $C_0 = 1$, $C_1 = C_0 \cdot C_0 = 1 \cdot 1 = 1$

Hence $C_n = C_{n-1} \cdot C_0 + C_{n-2} \cdot C_1 + C_{n-3} \cdot C_2 + \cdots + C_0 \cdot C_{n-1}$

$\Rightarrow C_2 = C_1 \cdot C_0 + C_0 \cdot C_1 = 2$

$C_3 = C_2 \cdot C_0 + C_1 \cdot C_1 + C_0 \cdot C_2 = 5$

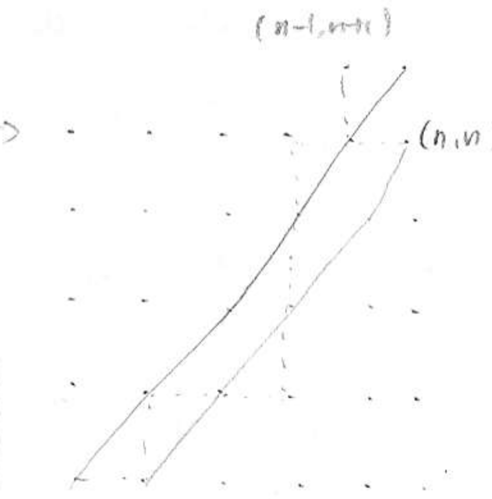
$\vdots$

$C_n = 1, 1, 2, 5, 14, 42 \ldots$

$14 \downarrow 5+2+2+5$ $\quad$ $42 \downarrow 14+5+4+5+14$

Method 2 : André's reflection

$\begin{pmatrix} \#\text{ good} \\ \text{path to } (n,n) \end{pmatrix} = (\#\text{ all paths}) - \begin{pmatrix} \#\text{ bad} \\ \text{paths} \end{pmatrix}$




$$= \binom{2n}{n} - \binom{2n}{n-1}$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{\frac{1}{n} \cdot n! \cdot (n+1)\,n!}$$

$$= \frac{(2n)!}{n!\,n!}\left(1 - \frac{n}{n+1}\right)$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

$$= C_n$$

$\Leftrightarrow (n+1)\,C_n = \binom{2n}{n}$

$\begin{pmatrix} \#\text{ of bad} \\ \text{paths to } (n,n) \end{pmatrix}$

$\uparrow \downarrow$ André's reflection

$\begin{pmatrix} \text{path to} \\ (n-1), (n+1) \end{pmatrix}$

Consider the first place a path hits the diagonal above $(k, k+1)$
reflect our path after that place about the given diagonal
The result is an ordinary soccer game landing at $(n-1, n+1)$

$$F = F_c = \sum_{n=0}^{\infty} C_n \cdot x^n$$

$$F^2 = (C_0 x^0 + C_1 x^1 + C_2 x^2 + \cdots)(C_0 x^0 + C_1 x^1 + C_2 x^2 + \cdots)$$

$$= (C_0 C_0) x^0 + (C_1 C_0 + C_0 C_1) x^1 + (C_2 C_0 + C_1 C_1 + C_0 C_2) x^2 + \cdots$$

$$= C_1 x^0 + C_2 x^1 + C_3 x^2 + \cdots$$

$$= \frac{1}{x} F - \frac{1}{x}$$

Moral: $F^2 = \frac{1}{x} F^2 - \frac{1}{x} \iff x F^2 - F + 1 = 0$

$$\Rightarrow F = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$+$ or $-$ ?

at $x_0$, $F(0) = 1$

RHS : $\frac{1 \pm 1}{2 \cdot 0} = \frac{0}{0}$

$$= \frac{1 - \sqrt{1-4x}}{2x}$$

Recall : For $n \in \mathbb{Z}_{\geq 0}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \text{ where } \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = ?$$

① $\Rightarrow$ simplify

$(a-x)(b-x) \cdots (z-x) = 0$ since $(x-x)$ exists somewhere between

Hence $\binom{n}{k} = 0$ (if $k > 0$)

So $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k$

② Define $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$ for any $\alpha \in \mathbb{R}$

Thm: $(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ ( For any $\alpha \in \mathbb{R}$, and if $|x|<1$ )

that is from above, $(1+x)^{\alpha}$ holds even though $n$ is not an integer.

pf: (half)

Reminder: Taylor's formula $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

Take $f(x) = (1+x)^{\alpha}$

$$f^{(1)}(x) = \alpha(1+x)^{\alpha-1} \longrightarrow f^{(1)}(0) = \alpha$$

$$f^{(2)}(x) = \alpha(\alpha-1)(1+x)^{\alpha-1} \longrightarrow f^{(2)}(0) = \alpha(\alpha-1)$$

$$f^{(3)}(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-2} \longrightarrow f^{(3)}(0) = \alpha(\alpha-1)(\alpha-2)$$

$$\vdots$$

$$f^{(k)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$$

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \cdot x^k$$

$$= \sum_{k\geq 0} \binom{\alpha}{k} x^k$$

(only for non-necessary integers)

$$\therefore \sqrt{1+y} = \sum_{k\geq 0}^{\infty} \binom{1/2}{k} y^k$$

Aside

$$\binom{1/2}{k} = \frac{1}{k!}\left(\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)\cdots\left(\tfrac{1}{2}-k+1\right)\right)$$

$$= \frac{1}{k!}\left(\tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)\cdots\left(-\tfrac{2k-3}{2}\right)\right)$$

$$= \frac{1}{k!}\cdot\frac{(-1)^{k-1}}{2^k}\cdot\underbrace{(1\cdot 3\cdot 5\cdots(2k-3))}_{(2k-3)!!}$$