Non-Homogeneous Equations by "Undetermined Coefficients"

examples

1. $Ly = y'' - 3y' - 4y = 2\sin x$. $\alpha_{1,2} = 4, -1$

Try $y = A\sin x + B\cos x$, get

$$y_1 = \frac{1}{17}(3\cos x - 5\sin x)$$

General solution,

$$y = \underset{\substack{\uparrow \\ \text{particular solution}}}{y_1} + \underbrace{c_1 e^{4x} + c_2 e^{-x}}_{\text{general solution of homogeneous equation}}$$

$Ax = b$

If x_1 is a solution then general solution is

$\{x_1 + x : Ax = 0\}$

$Ax_1 = b$

$A(x_1 + x) = b + 0 = b$

2. $y'' - 3y' - 4y = 4x^2$

Guess $y = Ax^2 + Bx + C$, get

$$2A - 3(2Ax + B) - 4(Ax^2 + Bx + C) = 4x^2$$

$$-4Ax^2 + \underbrace{(-4B - 6A)}_{0}x + \underbrace{2A - 3B - 4C}_{0} = 4x^2$$

$$A = -1, \quad B = \frac{3}{2}, \quad C = -\frac{13}{8}$$

$$y_{general} = -x^2 + \frac{3}{2}x - \frac{13}{8} + c_1 e^{4x} + c_2 e^{-x}$$

3. $y'' - 4y = xe^x + xe^{2x}$. $\alpha^2 - 4 = 0 \Rightarrow \alpha_{1,2} = \pm 2$

Part I ($y'' - 4y = xe^x$)

Guess $Axe^x + Be^x$

$\Rightarrow$ No problem!

$(xe^x)' = (x+1)e^x$

$(xe^x)'' = (x+2)e^x$

Part II ($y'' - 4y = xe^{2x}$)

① Guess Axe^{2x}, get

$y'' - 4y = 4e^{2x}$) Doesn't work

$(xe^{2x})' = (2x+1)e^{2x}$

$(xe^{2x})'' = (4x+4)e^{2x}$

② Guess $Ax^2e^{2x} + Bxe^{2x} + Ce^{2x}$

$\Rightarrow$ No problem!

In general, technique works if RHS is
(poly.) (exponential) + similar products
$\sum_{k=0}^{n} a_k x^k \cdot (e^{\alpha x})$

Guess will be
$(\sum_{k=0}^{n+m} A_k x^k) e^{\alpha x}$

m: multiplicity of α as a root of the original char. eqn.

RHS $e^{3x}\cos 2x = \dfrac{e^{(3+2i)x} + e^{(3-2i)x}}{2}$

Alternatively, if cos or sin appear in RHS, add them also on guess side.

Suppose RHS $= (x^2 - x - 1)e^x \cos 3x$
guess,
$\cdots + z_1 + (A_1 x^2 + B_1 x + C_1) e^x \cos 3x$
$\cdots + z_2 + (A_2 x^2 + B_2 x + C_2) e^x \cos 3x$

Systems of Constant Coefficient Homogeneous 1st order ODEs

$y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n$
$y_2' = \quad \vdots$
$\vdots$
$y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n$

$y(0) = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$

$A = (a_{ij}) \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$y' = Ay \quad A \in M_{n\times n}(\mathbb{R})$
$y(0) = y_0 \in \mathbb{R}^n$
Expect $y = e^{Ax} \cdot y_0$

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Baby Version

$A = (a)$ $\quad y_0 = (y_0)$

$y' = ay$ $\quad y(0) = y_0$

Solution ~~1.~~ $\cancel{y = y_0 e^{ax}}$

2. $y = e^{ax} y_0$

Reminder what e^x?

Def 1 $(e^x)' = e^x$

e^x is the sol. of $y' = y$

Def 2 $e^x = \sum \frac{x^k}{k!}$

Problem

Define e^{xA} when $A \in M_{n\times n}(\mathbb{R})$

1. $\longrightarrow$ solution of $(e^{xA})' = Ae^{xA}$

$\longrightarrow$ works but pointless

Aside

$M(x)$ a matrix depending on x

$$M(x)' = \lim_{h\to 0} \frac{M(x+h) - M(x)}{h}$$

$$= \lim_{h\to 0} \frac{(m_{ij}(x+h) - m_{ij}(x))_{ij}}{h} \quad \text{if } M(x) = (m_{ij}(x))$$

$$= \left(\lim_{h\to 0} \frac{m_{ij}(x+h) - m_{ij}(x)}{h}\right)_{ij}$$

$$= m_{ij}'(x)$$

2. **Definition**

If $A \in M_{n\times n}(\mathbb{R})$ s.t.

$$e^{xA} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k A^k$$

Theorem

1. converges
2. It has all good properties
3. It is fully computable