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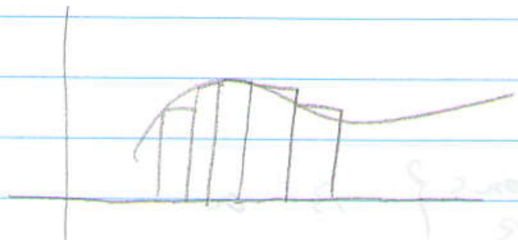
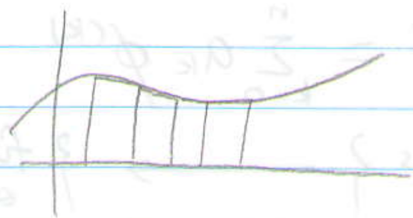
The special case $f(x,y) = f(x)$, $\phi' = f$, $\phi = \int f$, $\phi(a) = 0$

$\phi(x) = \int_a^x f(t)\,dt$

RK 4 $\Rightarrow$ an integration method



Euler Method



Improved Euler

$x_{n+1} = x_n + h$, $y_0 = 0$, $x_0 = a$

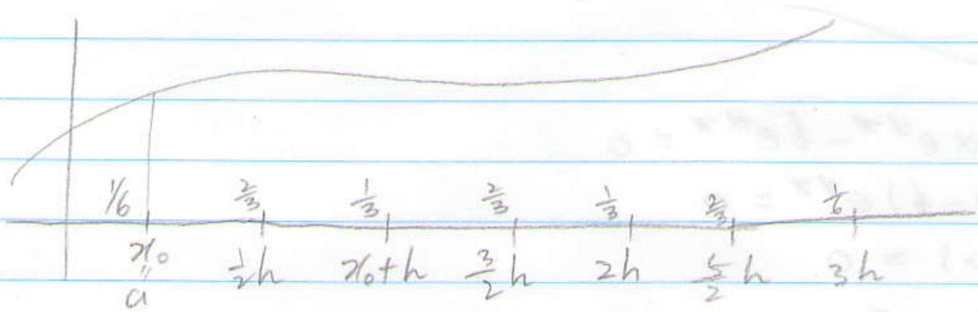
$k_1 = f(x_n, \cancel{y_n})$

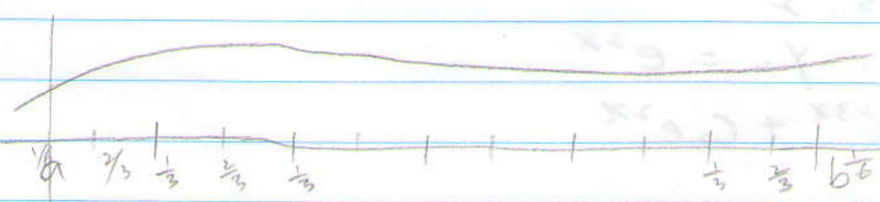
$k_2 = f(x_n + \frac{1}{2}h)$

$k_3 = f(x_n + \frac{1}{2}h)$

$k_4 = f(x_n + h)$

$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$





Simpson's Rule

$\int \doteq h \sum w_i f(\xi_i)$

Constant Coefficient Homogeneous High Order ODEs

$Ly = ay'' + by' + cy = 0 \quad a, b, c \in \mathbb{R}$

$Ly = \sum_{k=0}^{n} a_k y^{(k)} = 0 \quad a_k \in \mathbb{R}$

$L\phi = \sum_{k=0}^{n} a_k \phi^{(k)}$

$L : \{\text{functions on } \mathbb{R}\} \longrightarrow \{\text{functions on } \mathbb{R}\}$ is a

vector space → vector space

linear transformation "linear operator"

$y(0) = a_0$

$y'(0) = a_1$

$\vdots$

$y^{(n-1)}(0) = a_{n-1}$

What do we expect from $\{y : Ly = 0\} = \ker L$

→ expect an n-dimensional vector space

$y'' + y' - 6y = 0$ (guess $y = e^{\alpha x} \to y' = \alpha e^{\alpha x} \to y'' = \alpha^2 e^{\alpha x}$)

$\alpha^2 e^{\alpha x} + \alpha e^{\alpha x} - 6e^{\alpha x} = 0$

$(\alpha^2 + \alpha - 6)e^{\alpha x} = 0$

$(\alpha + 3)(\alpha - 2) = 0$

$\alpha_{1,2} = -3, 2$

$y_1 = e^{-3x}, \quad y_2 = e^{2x}$

$\therefore y = c_1 e^{-3x} + c_2 e^{2x}$

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$$0 = Ly = \sum_{k=0}^{n} a_k y^{(k)}$$

$$= \sum a_k D^k y$$

$$= P(D) \cdot y \quad \text{with } P(z) = \sum a_k z^k$$

D : {functions} $\rightarrow$ {functions}

$Df = \frac{d}{dx} f$

$D^k f = f^{(k)}$

Guess $y = e^{\alpha x}$, $Dy = \alpha y$

$0 = Ly = P(D)y = P(\alpha) y$

$P(\alpha) = 0$ in the lucky case, n distinct

n distinct real roots. $\alpha_1, \dots, \alpha_n \Rightarrow$ solutions are $e^{\alpha_i x}$

General solution is $\sum C_i e^{\alpha_i x}$

<u>If p has a complex root,</u>

<u>example</u>

$y'' - 4y' + 5y = 0$

$p(\alpha) = \alpha^2 - 4\alpha + 5 = 0$

$\alpha_{1,2} = 2 \pm i$

all coefficients are real

In a <u>real equation</u>, if $p(\alpha) = 0$, so is $P(\bar{\alpha})$

$y_1 = e^{(2+i)x} = e^{2x}(\cos x + i \sin x)$

$y_2 = e^{(2-i)x} = e^{2x}(\cos x - i \sin x)$

$$\frac{y_1 + y_2}{2} = e^{2x} \cos x = \phi_1$$

$$\frac{y_1 - y_2}{2i} = e^{2x} \sin x = \phi_2$$

If $\alpha = \beta + i\gamma \longrightarrow$ solutions $e^{\beta x} \cos \gamma x$, $e^{\beta x} \sin \gamma x$