Reminder: $f'(a;u) = \lim_{h\to 0} \dfrac{f(a+hu)-f(a)}{h}$ $\quad (f:\mathbb{R}^n \longrightarrow \mathbb{R}^m)$

$$\underbrace{f(\underbrace{a+h}_{\mathbb{R}^n})}_{\mathbb{R}^m} \underset{h \text{ small}}{\sim} \underbrace{f(a)+f'(a)\underbrace{h}_{\mathbb{R}^n}}_{\mathbb{R}^m}$$

$//L \qquad //R$

$f'(a): \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\quad f'(a) \in M_{m\times n}(\mathbb{R})$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a) \iff \lim_{h\to 0} \frac{f(a+h)-f(a)-f'(a)\cdot h}{h} = 0$$

$$\iff \lim_{h\to 0} \frac{L-R}{h} = 0$$

Def: $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ is diff. at $a\in\mathbb{R}^n$ if there exists a matrix $B\in M_{m\times n}(\mathbb{R})$ s.t. $f(a+h)\sim f(a)+B\cdot h$ for small $h\in\mathbb{R}^n$

1. Book's way

$$\frac{\|f(a+h)-f(a)-B\cdot h\|}{\|h\|} \xrightarrow[h\to 0]{} 0$$

2. Dror's way

$$o(h) = \{ \underset{\text{nbhd of } 0}{\phi: \mathbb{R}^n} \longrightarrow \mathbb{R}^m \mid \frac{\phi(h)}{h} \xrightarrow[h\to 0]{} 0 \}$$

$f(a+h)-f(a)-B\cdot h \in o(h)$

loosely. $f(a+h) = f(a)+B\cdot h + o(h)$

Thm : 1. If B exists, it is unique

Call it $Df(a)$ "the differential of f at a"

2. If f is constant, $(Df)(a) = 0_{m\times n}$

3. If f is linear, $f(x) = Ax$. $A \in M_{m\times n}$

$(Df)(a) = A$

4. $D(cf) = cDf$

$D(f+g) = Df + Dg$

5. If f is diff. $f'(a;u) = Df(a)\cdot u$,

$$Df(a) = \left(f'(a;e_1) \,\middle|\, f'(a;e_2) \,\middle|\, \cdots \right)$$

$f : \mathbb{R}^n_{x_i} \longrightarrow \mathbb{R}^m$

$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ $f_j = \mathbb{R}^n \longrightarrow \mathbb{R}$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

"Jacobian matrix of f"

pf: 1. Assume

$f(a+h) = f(a) + B_1 h + o(h)$

$\parallel$

$f(a+h) = f(a) + B_2 h + o(h)$

$(B_1 - B_2)h + (\phi_1 - \phi_2)$

$\phi_1, \phi_2 \in o(h)$

$o(h)$ is a v.sp.

$\rightsquigarrow (B_1 - B_2)h \in o(h)$

$$\frac{(B_1 - B_2)h}{\|h\|} \longrightarrow 0$$

$$B_1 - B_2 = 0$$

$$B_1 = B_2$$

2. $f(x) = c \in \mathbb{R}^m \quad \forall x$

$$f(x+h) = f(x) + B \cdot h + o(h)$$

$$c = c + 0 \cdot h + 0$$

$$\Rightarrow (Df)(a) = 0$$

3. $f(x) = Ax$

$$f(a+h) = A(a+h)$$

$$= Aa + Ah$$

$$= f(a) + Ah + 0$$

$$Df(a) = A$$

4. $(f+g)(a+h) \quad \ldots$

5. $f'(a; u) = \lim_{h \to 0} \dfrac{f(a+h \cdot u) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{f(a) + Df(a)h \cdot u + o(h \cdot u) - f(a)}{h}$$

$$= Df(a) \cdot u$$