Feb 10.

Thm. $\forall I \in \binom{n}{k}$ $\exists ! \psi_I \in A^k(V)$ s.t. $\forall J \in \binom{n}{k}$ $\psi_I(a_J) = \delta_{IJ}$ $\{\psi_I\}$ make a basis hence $\dim A^k(V) = \binom{n}{k}$

$*\mathbb{R}^3$ $I = (1,2)$ $\psi_I\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\right) = x_1y_2 - x_2y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. general.

Claim: In $V = \mathbb{R}^n$ with $a_i = e_i = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$; given $I \in \binom{n}{k}$.

$\psi_I(x_1 \dots x_k) = \det(X_I) = \det \begin{pmatrix} x_{1,i_1} & x_{2,i_1} & \cdots & x_{k,i_1} \\ x_{1,i_2} & & & x_{k,i_2} \\ \vdots & & & \vdots \\ x_{1,i_k} & \cdots & \cdots & x_{k,i_k} \end{pmatrix}$ where X_I indicates the matrix formed from rows $(i_1, i_2, \dots, i_k)$ of the matrix $X = (x_1 | \cdots | x_k)$.

Pf: Both sides are alternating & multi-linear in $x_1 \dots x_k$, so both sides are determined by their values on a_J, $J \in \binom{n}{k}$ $\quad \text{lhs}(a_J) = \psi_I(a_J) = \delta_{IJ}$.

$\text{rhs}(a_J) = \det(e_{j_1} | \cdots | e_{j_k})_I$.

$= \det \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{cases} \det(I_{k \times k}) = 1 & \text{if } I = J \\ \det\left(\substack{\text{some 0-1 matrix} \\ \text{with a row of 0}}\right) = 0 & \text{if } I \neq J \end{cases}$ $\square$

$\mathbb{R}^3$ $A^0 = \text{span}(\psi_{()})$ $\quad \psi_{()}() = 1$ $\quad 1$

$A^1 = \text{span}(\psi_{(1)}, \psi_{(2)}, \psi_{(3)})$ $\quad \psi_i\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_i$ $\quad 3$

$A^2 = \text{span}(\psi_{12}, \psi_{13}, \psi_{23})$ $\quad \psi_{13}\begin{pmatrix}x_1 & y_1\\x_2 & y_2\\x_3 & y_3\end{pmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}$ $\quad 3.$

$A^3 = \text{span}(\psi_{123})$ $\quad \psi_{23}(-) = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$

$A^4 = \{0\}$. $\quad \psi_{123}(x, y, z) = |x||y||z|$. $\quad 1$

$\otimes$ $\underset{f}{\mathcal{L}^k(V)} \times \underset{g}{\mathcal{L}^l(V)} \to \underset{f \otimes g}{\mathcal{L}^{k+l}(V)}$ $\quad (f \otimes g)(x_1 \dots x_{k+l}) = f(x_1 \cdots x_k)\, g(x_{k+1} \cdots x_{k+l})$

Suppose f, g are in A^k, A^l resp.

Is $f \otimes g \in A^{k+l}$? No. $\quad \underset{A^{k+l}}{(\underset{\cap\, A^1}{\psi_1} \otimes \underset{\cap\, A^1}{\psi_2})}(x, y) = x_1 y_2$.

Def. If $f \in A^k$, $g \in A^l$ $\quad (f \wedge g)(x_1 \cdots x_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (f \otimes g)^\sigma (-1)^\sigma$

$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} f(x_{\sigma_1} \cdots x_{\sigma_k}) \cdot (-1)^\sigma \cdot g(x_{\sigma_{k+1}} \cdots x_{\sigma_{(k+l)}})$

$= \sum_{\sigma \in S_{k+l}} (-1)^\sigma f(\cdots)\, g(\cdots)$ $\quad \sigma_1 < \sigma_2 < \cdots < \sigma_k$. $\sigma_{k+1} < \cdots < \sigma_{k+l}$.

In $A^*(\mathbb{R}^4)$ $\quad \underbrace{(\underset{\cap\, A^2}{\psi_{(13)}} \wedge \underset{\cap\, A^1}{\psi_4})}_{A^3}(x_1, x_2, x_3) = (\psi_{13})(x_1, x_2) \cdot \psi_4(x_3)$

$\sigma \in S_4$ have $\sigma_1 < \sigma_2$. $\quad - (\psi_{13})(x_1, x_3)\, \psi_4(x_2) + (\psi_{13})(x_2, x_3)\, \psi_4(x_1)$.

$\sigma_3 \dots$

Feb 13.

Read: Sec 27-29

Today: Wedge protects, target vectors. $f \in A^k(V)$, $g \in A^l(V)$, define $f \wedge g \in A^{k+l}(V)$ via

$(f \wedge g)(x_1 \dots x_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma f(x_{\sigma_1} \cdots x_{\sigma_k})\, g(x_{\sigma_{(k+1)}} \cdots x_{\sigma_{(k+l)}})$

$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma_1 < \cdots < \sigma_k \\ \sigma_{k+1} < \cdots < \sigma_{k+l}}} (-1)^\sigma f(x_{\sigma_1} \cdots x_{\sigma_k})\, g(x_{\sigma_{k+1}} \cdots x_{\sigma_{k+l}})$.

$f \wedge g(x_1, x_2, \dots x_5) = \frac{1}{3!\,2!} \sum_{\sigma \in S_5} \left(\substack{120 \\ \text{terms}}\right)$ $\quad f \in A^3(V)$ $\quad g \in A^2(V)$. $\quad \left|\left\{\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{matrix}\right\}\right| = \binom{5}{3} = 10$

$= \sum_{\substack{\sigma \in S_5 \\ \sigma_1 < \sigma_2 < \sigma_3 \\ \sigma_4 < \sigma_5}} \left(\substack{10 \\ \text{terms}}\right) = \cdots + (-1)^\sigma f(x_2, x_4, x_5) \cdot g(x_1, x_3) + \cdots$

Thm: $\exists!$ operation $\wedge \; A^k(V) \cdot A^l(V) \to A^{k+l}(V)$ s.t. $\in A^{k+l+m}$

1: $\wedge$ is bilinear & associative $(f+g)\wedge h = f\wedge h + g\wedge h$ --- $(f\wedge g)\wedge h = f\wedge(g\wedge h)$

2. $\wedge$ is super-symmetric $f\in A^k, g\in A^l$ $f\wedge g = (-1)^{k\cdot l}\, g\wedge f$

3. $\psi_I = \phi_{i_1}\wedge\phi_{i_2}\wedge\dots\wedge\phi_{i_k}$ $I=(i_1,\dots,i_k)$

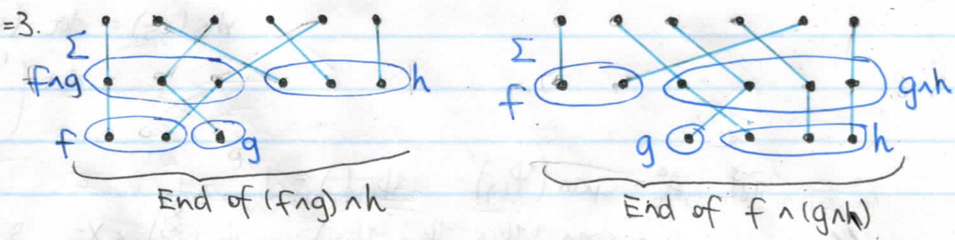
$e_1\cdot e_2 = e_2\cdot e_1$ $\quad e\cdot o = o\cdot e$. $\quad o_1\cdot o_2 = -o_2\cdot o_1$

Pf. (Existence) Take the $\wedge$ oper. defined before. NTS: Satisfies 1, 2, 3.

1: $(f\wedge g)\wedge h = f\wedge(g\wedge h)$ $\quad f\in A^k \quad g\in A^l \quad h\in A^m$

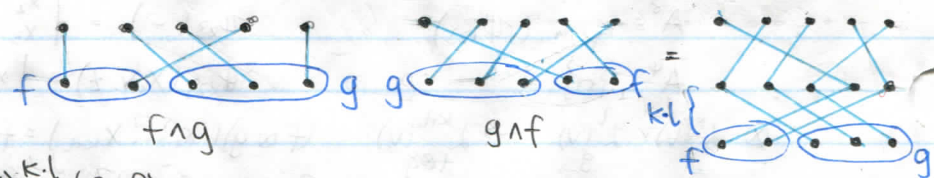
Need to feed both sides with $x_1, \dots x_{k+l+m}$ & see what they output.

$k=2 \quad l=1 \quad m=3$.



2. $(f\wedge g)(x_1,\dots x_{k+l})$ & $(g\wedge f)(x_1\dots x_{k+l})$.

$k=2 \quad l=3$



$f\wedge g = (-1)^{k\cdot l}(g\wedge f)$

3. $\psi_I = \phi_{i_1}\wedge\phi_{i_2}\wedge\dots\wedge\phi_{i_k}$ $\quad \psi_I(x_1\dots x_k) = \sum_{\sigma\in S_k}(-1)^k\phi_I(x_{\sigma_1},\dots x_{\sigma_k})$.

$= \sum_{\sigma\in S_k}(-1)^k\phi_{i_1}(x_{\sigma_1})\,\phi_{i_2}(x_{\sigma_2})\dots\phi_{i_k}(x_{\sigma_k})$

$=$ RHS.

(Uniqueness) Only one $\wedge$ satisfies 1. bil & assoc 2. super-symm 3. $\psi_I = \phi_{i_1}\wedge\dots\wedge\phi_{i_k}$

NTS. with 1, 2, 3 can compute $f\wedge g$ no matter what f & g are.

$f=\sum\alpha_I\psi_I \quad g=\sum\beta_J\psi_J$ Enough: $\psi_I\wedge\psi_J \overset{3}{=} (\phi_{i_1}\dots\phi_{i_k})\wedge(\phi_{j_1}\dots\phi_{j_l}) =$ sort.

$\overset{1}{=} \pm\psi_K$ for some K.

Feb 15.

Thm. $\exists!\ \wedge: A^k(V)\times A^l(V)\to A^{k+l}(V)$ s.t.

1. $\wedge$ is assoc. & bilinear.
2. $\wedge$ is super-symmetric $f\in A^k, g\in A^l \Rightarrow f\wedge g = (-1)^{k\cdot l}\, g\wedge f$
3. $\psi_I = \phi_{i_1}\wedge\phi_{i_2}\wedge\dots\wedge\phi_{i_k}$ if $I=(i_1,\dots i_k)\in\binom{n}{k}$.

only uniqueness remains.

Where $\phi_i(a_j)=\delta_{ij}$, $f=\sum_{I\in\binom{n}{k}}\alpha_I\psi_I \quad g=\sum_{J\in\binom{n}{l}}\beta_J\psi_J \quad (f\wedge g)=\sum_{I,J}\alpha_I\beta_J\,\psi_I\wedge\psi_J$

$\psi_I\wedge\psi_J \overset{3}{=} (\phi_{i_1}\wedge\phi_{i_2}\wedge\dots\wedge\phi_{i_k})\wedge(\phi_{j_1}\wedge\dots\wedge\phi_{j_l}) =$

$\overset{1}{=} \phi_{i_1}\wedge\dots\wedge\phi_{i_k}\wedge\phi_{j_1}\wedge\dots\phi_{j_l}$ $\quad\left(\overset{\text{Want}}{=} \psi_Q \quad Q\in\binom{n}{k+l}\right)$

$\overset{\text{repeatly use 2.}}{=} \pm\phi_{q_1}\wedge\phi_{q_2}\wedge\dots\wedge\phi_{q_{k+l}}$ $=$ ~~$\begin{cases}\pm\psi_Q & \text{if } q_1<q_2<\dots<q_{k+l} \text{ set } Q=(q_1,\dots q_{k+l})\in\binom{n}{k+l} \\ & \text{if } 7=q_4=q_5\end{cases}$~~

where $q_1\le q_2\le\dots\le q_{k+l}$

$$= \begin{cases} \pm \psi_Q & \text{if } q_1 < q_2 < \cdots < q_{k+l} \quad \text{set } Q = (q_1 \cdots q_{k+l}) \in \binom{n}{k+l}. \\ 0 & \text{if } 7 = q_4 = q_5 \quad (\cdots \wedge \phi_{q_4} \wedge \phi_{q_5} \wedge \cdots) = (\cdots \wedge \phi_7 \wedge \phi_7 \wedge \cdots) = -(\cdots \wedge \phi_7 \wedge \phi_7 \wedge \cdots) \end{cases}$$

Example: $\overset{k=2}{(\phi_2 \wedge \phi_5)} \wedge \overset{l=3}{(\phi_1 \wedge \phi_4 \wedge \phi_{257})} = (-1)^2 \phi_1 \wedge \phi_2 \wedge \phi_5 \wedge \phi_4 \wedge \phi_{257} = -\phi_1 \phi_2 \phi_4 \phi_5 \wedge \phi_{257}$

$(\phi_2 \wedge \phi_5) \wedge (\phi_1 \wedge \phi_5 \wedge \phi_{257}) = 0$.

* $V \xrightarrow{T} W \quad A^k(V) \xleftarrow{T^*} A^k(W) \quad T^*f \leftarrow f$

$(T^*f)(x_1 \cdots x_k) = f(Tx_1, \cdots, Tx_k). \quad x_i \in V$

Claim: $f \in A^k(W)$, $g \in A^l(W)$ $\quad \underbrace{\underset{A^k(V)}{(T^*f)} \wedge \underset{A^l(V)}{(T^*g)}}_{\in A^{k+l}(V)} = \underbrace{T^*(\underset{A^{k+l}(W)}{f \wedge g})}_{\in A^{k+l}(V)}$

Let $x_1 \cdots x_{k+l} \in V$ $\overset{\text{LHS}}{=} (T^*f) \wedge (T^*g)(x_1 \cdots x_{k+l}) = \frac{1}{k!} \sum (-1)^\sigma (T^*f)(x_{\sigma_1} \cdots x_{\sigma_k}) \cdot (T^*f)(x_{\sigma(k+1)} \cdots x_{\sigma_{k+l}})$

$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma f(Tx_{\sigma_1} \cdots Tx_{\sigma_k})\, g(\cdots \cdots)$

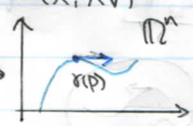
Def. A tangent vector ξ to $\mathbb{R}^n$ is a pair $\xi = (x, v)$ where $x, v \in \mathbb{R}^n$.

Def. The "tangent space to $\mathbb{R}^n$ at x" is $T_x(\mathbb{R}^n) = \{(x, v) : v \in \mathbb{R}^n\} \simeq \mathbb{R}^n$

$T_x(\mathbb{R}^n)$ is a vector space: 1. $\xi_1 + \xi_2 = (x, v_1 + x_2 v_2) := (x, v_1 + v_2)$. $\quad \xi_i = (x, v_i) \in T_x \mathbb{R}^n$

2. $\lambda(x, v) = (x, \lambda v)$

Get tangent at vectors: $\mathbb{R} \xrightarrow{\gamma}$



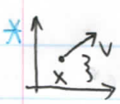
the tangent to a curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ (C^r) at $\gamma(p)$: $(\gamma(p), D\gamma_p(e_1))$

Do with tangent vectors.

Def. If $\xi \in T_x \mathbb{R}^n$ and $f : U \to \mathbb{R}$ where U is a nbd of x, assume f is diff'ble, the directional derivative of f in the direction of ξ to be $D_\xi f = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} = Df_x v$ where $\xi = (x, v)$

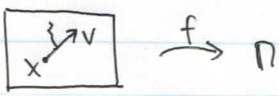
$\frac{d(f \circ \gamma)}{dt} = D_{(\gamma(t), D\gamma_t(e_1))} f$.

Feb. 17

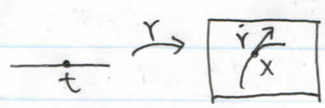


* Tangent vector $\xi = (x, v)$ $\quad T_x(\mathbb{R}^n) = \{\xi = (x, v)\}$ $\quad n$ vector space with ops acting only on the v-part.

$\dot\gamma(t) = (\gamma(t), D\gamma_t(e_1))$



$D_\xi f := Df_x(v)$

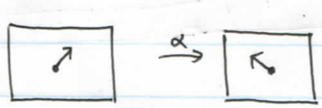


$\Rightarrow \frac{d(f \circ \gamma)}{dt} = D_{\dot\gamma(t)} f$

* $\alpha : \mathbb{R}^k \to \mathbb{R}^n$ $\quad$ standing assumption α is C^r

$(x, v) = \xi \in T_x \mathbb{R}^k \to (\alpha_* \xi) \in T_{\alpha(x)} \mathbb{R}^n$

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{(\alpha(x), D\alpha_x \cdot v)}$.

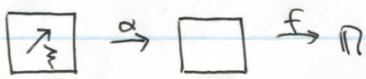


$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "\alpha"] \arrow[rr, bend left, "\beta\circ\alpha"] & \mathbb{R}^n \arrow[r, "\beta"] & \mathbb{R}^m \\ \xi \arrow[r] \arrow[rr, bend right, "(\beta\circ\alpha)_*\xi"'] & \alpha_*\xi \arrow[r] & \beta_*(\alpha_*\xi) \end{tikzcd}$$

Claim: $\beta_* \alpha_* \xi = (\beta\circ\alpha)_*(\xi)$

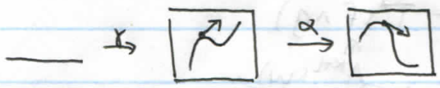
$\beta_* \circ \alpha_* = (\beta\circ\alpha)_*$

Pf: Chain Rule. $\square$.

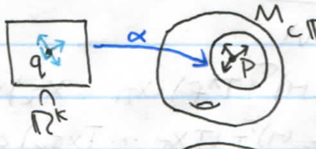


Claim: $D_\xi(\alpha^* f) = D_{\alpha_*\xi} f$

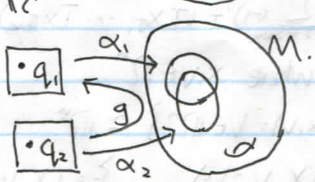
Pf: Chain Rule.



Claim: $\alpha_*(\dot\gamma(t)) = (\alpha\cdot\gamma)^{\boldsymbol\cdot}(t)$



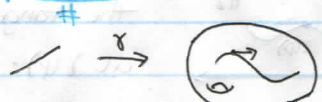
Def. $T_pM = \alpha_*(T_q\mathbb{R}^k) \subset T_p\mathbb{R}^n$

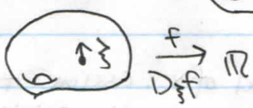


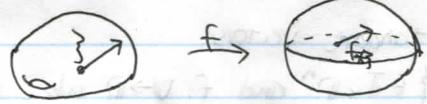
g is diff. # $(Dg)_{q_2}$ is invertible.

Claim: $\alpha_{1*} T_{q_1}\mathbb{R}^k = \alpha_{2*} T_{q_2}\mathbb{R}^k$

Pf: $\underbrace{\alpha_{1*} T_{q_1}\mathbb{R}^k}_{} = \alpha_{1*} \underbrace{g_* T_{q_2}\mathbb{R}^k}_{\#} = (\alpha_1 g)_* T_{q_2}\mathbb{R}^k = \alpha_{2*} T_{q_2}\mathbb{R}^k$ $\square$.

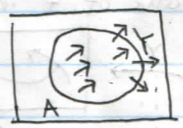
Comments: 1.  $\dot\gamma(t) \in T_{\gamma(t)} M$.

2. 

3. 

C^r vector fields on $A \subset \mathbb{R}^n$ $\quad Y: A \to \bigcup_{x\in A} T_x\mathbb{R}^n$ s.t.

$\forall x\in A \; Y(x) \in T_x\mathbb{R}^n$



In particular $Y(x) = (x, \sum y_i(x) e_i)$ $\quad Y \leftrightarrow$ function $\mathbb{R}^n \to \mathbb{R}^n$

$\{\text{functions on } A\} \xrightarrow{Y} \{\text{functions on } A\} \to (Yf)(x) = D_{Y(x)} f$.
$\quad \underset{F}{\Psi}$

Def. A v.f. Y is called C^r if 1. $\forall i.\ y_i$ is C^r $\quad$ 2. $\forall f \in C^{r+1}.\ Yf \in C^r$