$$\begin{vmatrix}1 & -1\\ -1 & 1\end{vmatrix} \overset{3}{=} \begin{vmatrix}1 & -1\\ 0 & 0\end{vmatrix} \underset{c=0}{\overset{2}{=}} 0 .$$

So prop. 0–3 determine det!

<u>Thm</u> A is invertible iff $\det(A) \neq 0$

<u>PF</u> compute the RREF B of A

using only 1–3, using 2 only w/

$c \neq 0$, $|A| = * |A_1| = ** |A_2| = \cdots$

$= \delta |B|$

δ is the product of factors encountered along the way.

$\delta = 1 (-1)(-1) \cdot 1 \cdot \underset{\neq 0}{c_1} \cdot \underset{\neq 0}{c_2} \neq 0$

If B is I then

$|A| = \delta |I| = \delta \neq 0$ & $\operatorname{rank} A = \operatorname{rank} I = n$

$\Rightarrow A^{-1}$ exists.

Otherwise B has a row of zeros;

$|A| = \delta|B| = 0$ & $\operatorname{rank} A = \operatorname{rank} B < n$

$\Rightarrow A^{-1}$ does not exist

$E^1_{i,j}\ E^2_{i,c}\ E^3_{i,j,c} = E$

$A \to EA$ performs an elem row ops

Thm If $A = E_1 E_2 E_3 \cdots E_k$ is a product of elem matrices then
$|A| = |E_1| \cdot |E_2| \cdots |E_k|$

<u>PF</u> prop 1-3 can be summarized as

$\det(EA) = |E| \det(A)$

So $\det A = \det(E_1 \overbrace{E_2 \cdots E_k}^{A'}) = |E_1| \det(A')$

$= |E_1| \det(E_2 \cdots E_k) = |E_1||E_2| \det(E_3 \cdots E_k)$

$= |E_1||E_2| \cdots |E_k|$

Thm For any $n \times n$ matrices A, B,

$$\det(A \cdot B) = \det(A)\det(B)$$

Pf: If $A \cdot B$ is not invertible either A is not invertible, or B isn't invertible.

In either case $|AB| \overset{\text{thm}}{\underset{1}{=}} 0 = |A||B|$

Otherwise AB is invertible, by HW, both A & B are invertible

So both A & B can be obtained from I by row ops.

So $A = E_1 E_2 \cdots E_k \cdot I = E_1 E_2 \cdots E_k$

$B = E_1' E_2' \cdots E_k'$

So $\det(AB) = \det(E_1 \cdots E_k E_1' \cdots E_k')$

$\overset{\text{thm2}}{=} \overbrace{|E_1||E_2| \cdots |E_k|}^{\text{thm2 } \det A} \overbrace{|E_1'| \cdots |E_k'|}^{\text{thm2 } \det B}$

$= \det(A) \cdot \det(B)$

$\square$