Def: $P' = (a = t_0' < t_1' < \cdots < t_{k'}' = b)$ is a "refinement" of

$P = (a = t_0 < t_1 < \cdots < t_k = b)$

if $\forall_j . \ t_j \in P'$ ($\forall j \ \exists j' \ \ t_j = t_{j'}'$). $\underline{P}' = (P_1', P_2', \cdots, P_n')$ "refines"

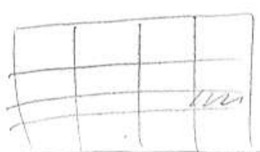
$\underline{P} = (P_1, \cdots, P_n)$ if $\forall_j \ P_j'$ is a refinement of P_j'

Lem: If $\underline{P}'$ refines $\underline{P}$, then ① $L(f, \underline{P}') \geq L(f, \underline{P})$ & ② $U(f, \underline{P}') \leq U(f, \underline{P})$

pf: ② May assume that $\underline{P}'$ is obtained from $\underline{P}$ by adding a single pt s on coord. # j_0 between $t_{j_0, i-1}$ & $t_{j_0, i}$

so $P_j = P_j'$ for $j \neq j_0$ &

$$P_{j_0}' = (a_{j_0} = t_{j_0,0} < t_{j_0,1} < \cdots < t_{j_0, i-1} < s < t_{j_0, i} < \cdots < t_{j_0, k_{j_0}} = b)$$



Enough to consider rectangles R' in $\underline{P} = (P_1, \cdots, P_n)$ of the form $R' = \prod_{j=1}^{j_0 - 1} J_j \times [t_{j_0, i-1}, t_{j_0, i}] \times \prod_{j = j_0 + 1}^{n} J_j \quad \forall_j \neq j_0 \quad J_j \in P_j$

After refinement, namely, in $\underline{P}'$ $\quad R' \longrightarrow R_1 \cup R_2$ where

$$R_1 = \prod_{j=1}^{j_0 - 1} J_j \times [t_{j_0, i-1}, s] \times \prod_{j = j_0+1}^{n} J_j$$

$$R_2 = \prod_{j=1}^{j_0 - 1} J_j \times [s, t_{j_0, i}] \times \prod_{j = j_0+1}^{n} J_j$$

$$V(R') = V(R_1) + V(R_2)$$

$$\underbrace{\Big(\prod_{j \neq j_0} |J_j|\Big)}_{A} (t_{j_0, i} - t_{j_0, i-1}) \overset{?}{=} A(s - t_{j_0, i-1}) + A(t_{j_0, i} - s)$$

$M_{R'}(f) \geq M_{R_1}(f)$ & $M_{R'}(f) \geq M_{R_2}(f)$ $\Rightarrow$ Obvious as $R' \supset R_1, R_2$

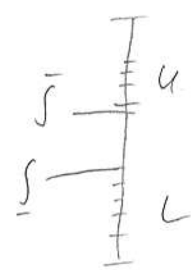
$$\Rightarrow M_{R'}(f) \cdot v(R') = M_{R'}(f)(v(R_1) + v(R_2))$$
$$\geq M_{R_1}(f)\, v(R_1) + M_{R_2}(f)\, v(R_2)$$

$\Rightarrow U(f, \underline{P}) \geq U(f, \underline{P}')$.

Likewise $L(f, P) \leq L(f, \underline{P}')$

<u>Lem</u>: For any $\underline{P}$ & $\underline{P}'$, $L(f, \underline{P}) \leq U(f, \underline{P}')$

~~pf~~: if P_1 & P_2 are partitions of $[a, b]$, find P_3 that refines both



$P_3 = P_1 \vee P_2$

Likewise. $\underline{P}'' = (P_1 \vee P_1', P_2 \vee P_2', \cdots)$

$\underline{P}''$ refines both $\underline{P}$ & $\underline{P}'$

Now $L(f, \underline{P}) \leq L(f, \underline{P}'')$
$\leq U(f, \underline{P}'')$
$\leq U(f, \underline{P}')$ $\square$

<u>Cor</u>: $\underline{\int}_Q f$ and $\bar{\int}_Q f$ make sense & $\underline{\int} f \leq \bar{\int} f$ $\square$

<u>Prop</u>: (The Riemann Condition)

f is integrable iff $\forall \varepsilon > 0$ $\exists \underline{P}$ s.t. $U(f, \underline{P}) - L(f, \underline{P}) < \varepsilon$