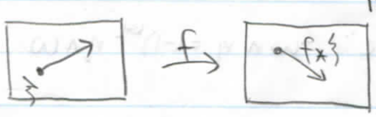
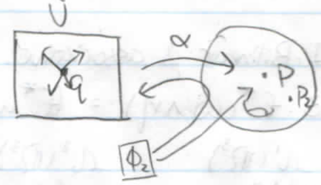
Feb 27



$T_x\mathbb{R}^n$

$\xi = (x, v)$

$f_*\xi = (f(x), Df_x v)$



$\mathbb{R}^k$, α, ϕ_2, $\subset \mathbb{R}^n$

$T_pM := \alpha_* T_q\mathbb{R}^k$

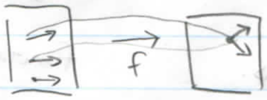
C^r vector fields: $Y: A\subset\mathbb{R}^n \to \bigcup_{x\in A} T_x\mathbb{R}^n$ s.t. $Y(x)\in T_x\mathbb{R}^n$ so $Y(x) = \sum Y_i(x)\cdot e_i$

f function → Yf by $(Yf)(x) = D_{Y(x)}f = \sum Y_i(x)\frac{\partial f}{\partial x_i}$ Y is C^r means $\forall i\ Y_i\in C^r \iff (f\in C^r \Rightarrow Yf\in C^r)$.

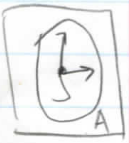
Aside: $x_j:\mathbb{R}^n\to\mathbb{R}$, $\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}\mapsto \alpha_j$ "Coordinate fcn"

Pf of above equivalence: (⇒) Assume $Y_i\in C^r$; $f\in C^{r+1}$ $Yf = \sum \underbrace{\underbrace{Y_i(x)}_{C^r}\cdot\underbrace{\frac{\partial f}{\partial x_i}}_{C^r}}_{C^r} \in C^r$ □.

(⇐) Take $f = x_j$, $C^r \ni Yf = \sum_{i=1}^n Y_i(x)\frac{\partial x_j}{\partial x_i} = Y_j(x)$ □



vert. field don't push or pull.



Def. A "k-tensor field" on open $A\subset\mathbb{R}^n$ is a map $\omega: A\to \bigcup_{x\in A}\mathcal{L}^k(T_x\mathbb{R}^n)$ = miss.

s.t. $\omega(x)\in\mathcal{L}^k(T_x\mathbb{R}^n)$

If $\xi_1\dots\xi_k\in T_x\mathbb{R}^n$ $\xi_i=(x,v_i)$ $\omega(\xi_1,\dots\xi_k)=\omega(x)(\xi_1\dots\xi_k)$

If $Y_1\dots Y_k$ are v.field on A, $\omega(Y_1\dots Y_k)(x) = \omega(x)(Y_1(x),\dots,Y_k(x))$ is a fcn on A.

Def. A k-form on $A\subset\mathbb{R}^n$ is a k-tensor field ω, s.t. $\omega(x)\in A^k(T_x\mathbb{R}^n)$

Suppose ω is either a k-tensor field or a k-form on $A\subset\mathbb{R}^n$

k-tensor field. $\omega(x) = \sum_{I\in[n]^k} a_I(x)\Phi_I$ where $\Phi_I\sim\phi_I$ on $\mathbb{R}^n$ meaning $\Phi_I(\xi_1\dots\xi_k) = \phi_I(v_1\cdots v_k)$

k-form: $\omega(x)=\sum_{I\in\binom{n}{k}} a_I(x)\psi_I$ $\xi_i=(x,v_i)$.

Def-thm: W is called C^r if 1: When writing $\omega=\sum a_I\phi_I$ or $\sum a_I\psi_I$ $\forall I.\ a_I\in C^r$

⟺ $\omega(Y_1\cdots Y_k)$ is C^r whenever $Y_1\cdots Y_k$ are C^r vector fields.

Def. $f:\mathbb{R}^n\to\mathbb{R}^m$ & ω is a k-tensor field on $\mathbb{R}^m$ then

k-tensor field on $\mathbb{R}^n$ ∋ $f^*\omega(\xi_1\dots\xi_k) = \omega(f_*\xi_1,\dots f_*\xi_k)$ where $\xi_1\dots\xi_k\in T_x\mathbb{R}^n$

If ω is a k-form, $f^*\omega$ is too.

1: $f^*g^*\omega = (g\circ f)^*\omega$ 2. $f^*(a\omega_1+b\omega_2) = af^*(\omega_1)+bf^*(\omega_2)$

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k-form on M. $\omega: M\to\bigcup A^k(T_pM)$ s.t. $\omega(p)\in A^k(T_pM)$ on $\mathbb{R}^n$, $\omega=\sum_{I\in\binom{n}{k}} a_I(x)\psi_I = \sum_I a_I(x)\phi_{i_1}\wedge\cdots\wedge\phi_{i_k}$

ω is C^r means: $\forall I$ $a_I\in C^r \iff \forall C^r$ v.f. $Y_1\cdots Y_k$, $\omega(Y_1\cdots Y_k)$ is C^r. forms pullback:

$f:\mathbb{R}^n\to\mathbb{R}^m$, ω on $\mathbb{R}^m$ $\xi_1\dots\xi_k\in T_x\mathbb{R}^n$. $(f^*\omega)(\xi_1\cdots\xi_k) = \omega(f_*\xi_1,\dots,f_*\xi_k)$

Def: $\Omega^k(\mathbb{R}^n)/\Omega^k(M)$ the vector space of all C^∞ k-forms on $\mathbb{R}^n$/M.

$(\omega_1+\omega_2)(\xi_1\dots\xi_k) = \omega_1(\xi_1\cdots\xi_k)+\omega_2(\xi_1\cdots\xi_k)$

$(\lambda\omega)(\xi_1\dots\xi_k) = \cdots\cdots$

$\wedge: A^k(V)\times A^l(V)\to A^{k+l}(V)$.

Def. $\omega\in\Omega^k(M)$ $\eta\in\Omega^l(M)$ $(\omega\wedge\eta)(\xi_1\dots\xi_{k+l}) = (\omega(x)\wedge\eta(x))(v_1\dots v_{k+l})$

$\Omega^{k+l}(M)$ $\xi_i=(x,v_i)$ $v_i\in T_xM$.

properties: 1. Bilinear & associative. 2. "Super-symmetric" $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$

3. $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$

$\boxed{\text{On } \mathbb{R}^3}$

$\Omega^1(\mathbb{R}^3)$	$\Omega^2(\mathbb{R}^3)$	$\Omega^3(\mathbb{R}^3)$	$\Omega^0(\mathbb{R}^3)$
$\left\{ \begin{array}{l} a_1(x)\phi_1 \\ + a_2(x)\phi_2 \\ + a_3(x)\phi_3 \end{array} \right\}$	$\left\{ \begin{array}{l} b_1(x)\phi_2\wedge\phi_3 \\ + b_2(x)\phi_3\wedge\phi_1 \\ + b_3(x)\phi_1\wedge\phi_2 \end{array} \right\}$	$\{c(x)\phi_1\wedge\phi_2\wedge\phi_3\}$	$\{f(x)\}$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$a(x) = \begin{pmatrix} a_1(x) \\ a_2(x) \\ a_3(x) \end{pmatrix}$	$b(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ b_3(x) \end{pmatrix}$	$c(x)$	
IIS VF	IIS VF	IS functions.	function

0-form on M: $\omega : M \to \bigcup_{p\in M} \overbrace{A^0(T_pM)}^{\mathbb{R}}$ $\quad \omega: M \to \mathbb{R}$. $\iff$ a fcn on M.

ω is a 0-form and η is a k-form.

$(\omega \wedge \eta)(\xi_1 \cdots \xi_k) = \omega(x) \cdot \eta(x)(v_1 \cdots v_k)$.

$d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$

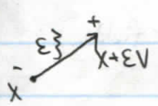
"Right def" (but too hard) Suppose $\omega \in \Omega^k(\mathbb{R}^n)$ $\quad (d\omega)(\xi_1 \cdots \xi_{k+1}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{k+1}} \underbrace{\omega(\partial P(\epsilon\xi_1, \dots \epsilon\xi_{k+1})}_{\text{Sum with signs of evals of } \omega \text{ on } k \text{ vectors}})$

$k=2$.

$= P(\xi_1 \cdots \xi_3)$ $\quad 2(k+1)$ faces.

∂P = Union of $2(k+1)$ faces.

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.



Following def. Let $f \in \Omega^0(\mathbb{R}^n)$ ~ f is a function. what is $(\partial F)(\xi) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(f(x+\varepsilon v) - f(x))$

$\Omega^1(\mathbb{R}^n)_{\xi = (x,v)} = D_\xi f$

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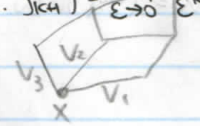
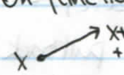
$\int_M d\omega = \int_{\partial M} \omega$ $\quad d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ by $d\omega(\xi_1 \dots \xi_{k+1}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{k+1}} \omega(\partial P(\varepsilon\xi_1, \epsilon\xi_{k+1}))$

where $P(\xi_1, \xi_2, \xi_3)$ $\quad \xi_i = (x, v_i)$



On functions $(df)(\xi) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(f(x+\varepsilon v) - f(x)) = D_\xi f = Df_x \cdot v$.



$k=0$. Aside $(dx_j)(x, e_i) = \frac{\partial x_j}{\partial x_i} = \delta_{ij} = \phi_j(x, e_i)$ $\Rightarrow dx_j = \phi_j$

$x_j : \mathbb{R}^n \to \mathbb{R}$. "the coord. fcns" on $\mathbb{R}^m$.

Claim. If $f \in C^1(\mathbb{R}^n)$ $\quad df = \sum \frac{\partial f}{\partial x_i} \phi_i = \sum_i \frac{\partial f}{\partial x_i} dx_i$ $\quad$ (on $\mathbb{R}^1$, $df = \frac{df}{dx} \cdot dx$

pf: eval both sides on (x, e_j) $\quad$ LHS $= df(x, e_j) = D_{e_j} f(x) = \frac{\partial f}{\partial x_j}$

RHS $= \sum_i \frac{\partial f}{\partial x_i}(x)\,(dx_i)(e_j) = \sum \frac{\partial f}{\partial x_i} \delta_{ij}$.

functions $\xrightarrow[\nabla f]{\text{grad}(f)}$ vector fields $\quad f \to \partial f/\partial x_1\, dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 \leftrightarrow \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix}$

Thm: $\exists!$ linear operator $d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ s.t.

1. If $f \in \Omega^0(\mathbb{R}^n)$ $df = \sum \frac{\partial f}{\partial x_i} \cdot dx_i$ 2. $\omega \in \Omega^k, \eta \in \Omega^l, d(\omega \wedge \eta) = (d\omega)\wedge\eta + (-1)^k \omega \wedge (d\eta)$

3. $d^2 = 0$: $d \circ d = 0$ if $\omega \in \Omega^k(\mathbb{R}^n)$ then $d(d\omega) = 0$ in $\Omega^{k+2}(\mathbb{R}^n)$

$$\boxed{\begin{array}{l} dx_I = dx_{i_1}\wedge\cdots\wedge dx_{i_k} \\ d(dx_I) = d(dx_{i_1}\wedge\cdots\wedge dx_{i_k}) \\ = 0. \end{array}}$$

$$d\omega = d\Big(\sum_{I \in \binom{n}{k}} a_I(x)\cdot dx_I\Big) = \sum_I d(a_I\, dx_I) = \sum_I (da_I)\wedge dx_I \cancel{\pm \sum a_I \wedge d(dx_I)}$$

$$= \sum_I \Big(\sum_i \frac{\partial a_I}{\partial x_i} dx_i\Big)\wedge dx_I = \sum_{i,I} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I.$$

$\Rightarrow$ we have prove uniqueness in thm!

$$d(b_1 dx_2\wedge dx_3 + \cdots) = (db_1)\wedge dx_2 \wedge dx_3 + \cdots = \Big(\frac{\partial b_1}{\partial x_1}dx_1 + \frac{\partial b_1}{\partial x_2}dx_2 + \frac{\partial b_3}{\partial x_3}dx_3\Big) dx_2\wedge dx_3 + \cdots$$

$$= \frac{\partial b_1}{\partial x_1} dx_1\wedge dx_2 \wedge dx_3 + \frac{\partial b_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \cdots$$

$$= \Big(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}\Big) dx_{123}$$

$$\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\{f(x)\} & & \left\{\begin{array}{l} a_1(x)\,dx_1 \\ + a_2(x)\,dx_2 \\ + a_3(x)\,dx_3 \end{array}\right\} & \longrightarrow & \left\{\begin{array}{l} b_1(x)\,dx_2\wedge dx_3 \\ + b_2(x)\,dx_3\wedge dx_1 \\ + b_3(x)\,dx_1\wedge dx_2 \end{array}\right\} & & \{c(x)\,dx_1\wedge dx_2\wedge dx_3\} \\
\text{functions} & \xrightarrow[\nabla f]{\text{grad}(f)} & \text{vector fields} & \xrightarrow[\nabla\times]{\text{curl}} & \text{vector fields.} & \xrightarrow[\nabla\cdot B]{\text{div} B} & \text{functions} \\
f & \longrightarrow & \begin{array}{l} (\partial f/\partial x_1)\,dx_1 \\ + (\partial f/\partial x_2)\,dx_2 \\ + (\partial f/\partial x_3)\,dx_3 \end{array} \Leftrightarrow \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{pmatrix} & & \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = B. & & \sum \frac{\partial b_i}{\partial x_i}
\end{array}$$

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Thm: $\exists!$ linear $d: \Omega^k \to \Omega^{k+1}$ s.t. 1: $f \in \Omega^0 \Rightarrow (df)(\xi) = D_\xi f = \sum_i \frac{\partial f}{\partial x_i} dx_i$

2. $d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^{\deg \omega}\, \omega \wedge d\eta$. 3. $d^2 = 0$.

pf: 1-3. imply $d(\sum a_I dx_I) = \sum_{j,I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$ proves uniqueness.

$$\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\
\{f\}. & & \{a_1 dx_1 + a_2 dx_2 + a_3 dx_3\} & & \{b_1 dx_2\wedge dx_3 + \text{cyc perms}\} & & \{c\,dx_1\wedge dx_2\wedge dx_3\}. \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\{f\} & \xrightarrow[\nabla]{f \mapsto \text{grad}(f)} & \left\{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right\} & & \left\{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right\} & \xrightarrow[\nabla\cdot B]{B \mapsto \text{div} B} & \{c\} \\
f & \longmapsto & \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{pmatrix} & & B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} & \longrightarrow & \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}
\end{array}$$