Thm: A bdd $f: Q \to \mathbb{R}$ is integrable iff its discts set is of msr 0.

Pf: Assume $|f(x)| \le M \quad \forall x \in Q$.

($\Leftarrow$) Assume $D(f)$ is of msr 0.

Let $\varepsilon > 0$ be given (for the Riemann cty)

Find a ctble collection Q_i of rectangles s.t.

$D(f) \subset \bigcup \operatorname{int} Q_i$ & $\sum v(Q_i) < \varepsilon_1$

For each $a \in Q \setminus D(f)$, find a rectangle Q_a s.t. $a \in Q_a$ &

$$M_Q(f) - m_Q(f) = \sup\{ f(x) \mid x \in Q_a\} - \inf\{ f(x) \mid x \in Q_a\} < \varepsilon_2$$

Now $\{\operatorname{int} Q_i\} \cup \{\operatorname{int} Q_a\}$ covers Q

Find using cptness of Q, a fin. subcvr $\{\operatorname{int} Q_1', \dots, \operatorname{int} Q_n'\} \subset \{\operatorname{int} Q_i\}$ & $\{\operatorname{int} Q_1'', \dots, \operatorname{int} Q_m''\} \subset \{\operatorname{int} Q_a\}$

$$Q = \bigcup_{i=1}^{n} Q_i' \cup \bigcup_{i=1}^{m} Q_i''$$

Let $\underline{P}$ be a partition of Q s.t. each Q_i' & each Q_i'' is a union of rectangles in $\underline{P}$.

Now $U(f, \underline{P}) - L(f, \underline{P}) = \sum_{R \in \underline{P}} v(R)[M_R(f) - m_R(f)]$

$$\le \underbrace{\sum_{\substack{R \in \underline{P} \\ R \subset \cup Q_i'}} \text{same}}_{A} + \underbrace{\sum_{\substack{R \in \underline{P} \\ R \subset \cup Q_i''}} \text{same}}_{B} \qquad (\#)$$

$$A = \sum_{\substack{R \in \underline{P} \\ R \subset \cup Q_i'}} v(R)\,(M_R(f) - m_R(f))$$

$$\leq \sum_{\substack{R \in \underline{P} \\ R \subset \cup Q_i'}} v(R)\, 2M$$

$$= 2M \sum_{\substack{R \in \underline{P} \\ R \subset \cup Q_i'}} v(R)$$

$$= 2M \sum v(Q_i')$$

$$< 2M \cdot \varepsilon_1$$

$$B = \sum_{\substack{R \in \underline{P} \\ R \subset \cup Q_i''}} v(R)\,\underbrace{(M_R(f) - m_R(f))}_{< \varepsilon_2}$$

$$< v(Q) \cdot \varepsilon_2$$

$$\# = A + B$$

$$< \underbrace{2M\varepsilon_1}_{\varepsilon/2} + \underbrace{v(Q)\varepsilon_2}_{\varepsilon/2}$$

$$= \varepsilon$$

Changing the beginning of the pf by setting

$$\varepsilon_1 = \frac{\varepsilon}{4M} \quad \text{and} \quad \varepsilon_2 = \frac{\varepsilon}{2v(Q)}$$

□

($\Rightarrow$) (Sketch)

Assume f is integrable.

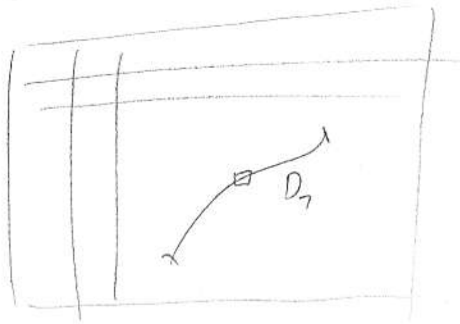
Define "oscillation of f at a" $= v(f, a) = \inf (M_R(f) - m_R(f))$

"Jump of f at a"

Claim: $v(f,a) > 0 \iff a \in D(f) \Rightarrow D(f) = \bigcup_{m=1}^{\infty} D_m$

$$D_m = \{ a \mid v(f,a) \geq \tfrac{1}{m} \}$$

Pf: Enough to show that each D_m is of msr 0.



Find P s.t. $U(f,P) - L(f,P) < \frac{\varepsilon}{7}$

$$\sum_{R \in P} v(R)(M_R(f) - m_R(f)) \geq \sum_{\substack{R \in P \\ R \cap D_7 \neq \emptyset}} v(R)(M_R(f) - m_R(f))$$

$$\geq \sum v(R) \frac{1}{7}$$

mult. by 7

$$\Rightarrow \sum_{R \cap D_7 \neq \emptyset} v(R) < \varepsilon$$