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The non-homogeneous case

$\dot{v}(t) = Av(t) + g(t)$

Two approaches

1. Use diagonalization

Set $v = cu$ where $u = u(t) \in \mathbb{R}^n$ & $c \in M_{n\times n}(\mathbb{R})$

$$\begin{aligned} \dot{v} &= (cu)' \\ &= c\dot{u} \\ &= Acu + g \\ \therefore\ \dot{u} &= c^{-1}Acu + c^{-1}g \end{aligned}$$

If $C^{-1}AC = D$ is diagonal, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

$\dot{u} = Du + c^{-1}g$. This is a de-coupled system

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \\ \vdots \\ \lambda_n u_n \end{pmatrix} + \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{pmatrix} \qquad c^{-1}g(t) = h(t) \in \mathbb{R}^n$$

$$\Leftrightarrow \begin{cases} \dot{u}_1 = \lambda_1 u_1 + h_1 \\ \dot{u}_2 = \lambda_2 u_2 + h_2 \\ \vdots \\ \dot{u}_n = \lambda_n u_n + h_n \end{cases}$$

"de-coupling"
We know how to solve!

Example

$$\dot{v} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} v + \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}$$

Eigenvalues of $A = 0, -5$

Eigenvectors of $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \quad C^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix}$$

$$\dot{u} = Du + \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -5u_2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 5t^{-1}+8 \\ 4 \end{pmatrix}$$

$$\dot{u}_1 = \frac{1}{5}(5t^{-1}+8). \quad \dot{u}_2 = -5u_2 + \frac{4}{5}$$

$$u_1 = \log t + \frac{8}{5}t + C_1$$

$$u_2 = \frac{4}{25} + C_2 e^{-5t}$$

$$v = Cu$$

$$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \log t + \frac{8}{5}t + C_1 \\ \frac{4}{25} + C_2 e^{-5t} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -\frac{8}{25} + \frac{8}{5}t + \log t \\ \frac{4}{25} + \frac{16t}{5} + 2\log t \end{pmatrix}}_{\text{A particular solution of the full equation}} + \underbrace{C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{\text{The general solution of the homogeneous equation}}$$

$$v' = Av + g \iff v' - Av = g$$
$$\iff Lv = g$$

If $C^{-1}AC = D = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$\dot{u} = Du + h \quad , \quad h = C^{-1}g$

$\dot{u}_1 = \lambda u_1 + u_2 + h_1$

$\dot{u}_2 = \lambda u_2 + h_2 \quad \rightarrow$ Solve this on first. find u_2, substitute into 1st equation & solve as usual.

2. Using "Fundamental Matrix"

$\dot{v} = A(t)v(t) + g(t)$

Suppose we have n linearly independent solutions $\varphi_1, \dots, \varphi_n$ of the homogeneous equation : $\dot{\varphi}_i = A(t)\varphi_i(t)$

A "fundamental matrix" is the matrix

$$\psi = (\varphi_1 | \varphi_2 | \varphi_3 | \cdots | \varphi_n)$$

satisfying $\dot{\psi} = A\psi$

Claim

Given $\dot{\psi} = A\psi$. ψ is either regular (invertible) for all t or singular for all t

(in our case, as $\varphi_1, \dots, \varphi_n$ where independent, there is some t in which ψ is regular, therefore it is always regular)

$(\psi u)' = A\psi u + g$

$\psi' u + \psi \dot{u} = A\psi u + g$

$A\psi u + \psi \dot{u} = A\psi u + g$

$\psi \dot{u} = g$

$\dot{u} = \psi^{-1} g$

$\therefore u = \int_{t_0}^{t} \psi^{-1} g \, dt \Rightarrow V = \psi \int \psi^{-1} g \, dt$

Example

$$\dot{v} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} v + \begin{pmatrix} t^{-1} \\ 2t^{-1}+4 \end{pmatrix}$$

Solutions of $\dot{v} = av$ are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\psi = \begin{pmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{pmatrix} \quad \psi^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2e^{-5t} & e^{5t} \end{pmatrix}$$

$$\psi^{-1} g = \begin{pmatrix} \frac{8}{5} + t^{-1} \\ \frac{4e^{5t}}{5} \end{pmatrix} \Rightarrow \int \psi^{-1} g = \begin{pmatrix} \frac{8}{5}t + \log t \\ \frac{4}{25}e^{5t} \end{pmatrix}$$

$\psi \cdot \int \psi^{-1} g = \cdots =$ same as before

Proof of claim

1. Only use Fundamental Theorem.
Suppose ψ is singular at t_0
$\Leftrightarrow$ columns of ψ are linearly dependent at t_0
$\Leftrightarrow$ $\exists$ constants not all 0, a_i, s.t. $\sum a_i \psi_i(t_0) = 0$
Consider $\phi(t) = \sum a_i t_i(t)$

Then ϕ solves $\dot{\phi} = A\phi$ & also $\phi(t_0) = 0$
but 0 also solve $\dot{0} = A0$, $0(t_0) = 0$
By uniqueness $0 = \phi$ for all t, so $\sum a_i \psi_i(t) = 0$
$\therefore \psi$ is everywhere singular.