The extreme value thm: A cts fct on a cpt sp attains its inf and its sup

Def: X is "conn" if the only clopen sets in X are ϕ and X

e.g: $\mathbb{R} \setminus \{257\}$


257

Not conn.: $(-\infty, 257) = (-\infty, 257] \cap X$ is clopen.

$\Leftrightarrow$ $X = A \dot\cup B$, $A \cap B = \phi$ A & B are both open (or both closed)

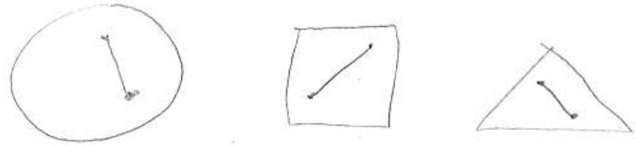
at least one of A or B is empty

Thm1: A subset X of $\mathbb{R}$ is conn. iff it is a "general" interval

$[a, b]$, $(-\infty, a]$, $(-\infty, \infty)$, ϕ

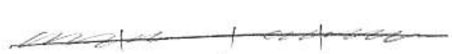
$\Leftrightarrow$ $\forall a, b \in X$. $[a, b] \subset X$


X is "convex"



pf: ($\Rightarrow$) Assume X conn. yet not convex.

Namely $\exists a, b, c$ s.t. $a < b < c$ $a, c \in X$ yet $b \notin X$


a b c

Then $X = \underbrace{(X \cap (-\infty, b))}_{= A'} \dot\cup \underbrace{(X \cap (b, \infty))}_{= B}$

A & B are open. $a \in A$, $b \in B$

So X not conn

($\Leftarrow$) I'll only prove that $[0,1]$ is conn.

Assume $[0,1] = A \dot{\cup} A^c$ where A clopen & non-trivial.

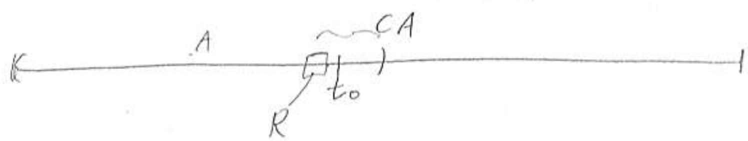
WLOG, $0 \in A$

Let $t_0 = \sup\{t \in [0,1] \mid [0,t] \in A\} = R$

Then $t_0 \in A$ as A closed.

But A is open. So for some $\varepsilon > 0$,

$(t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,1] \subset A$



By def of R, find $r \in R$ s.t. $r \in (t_0 - \varepsilon, t_0)$

But then $\underbrace{[0,r)}_{\subset A} \cup \big(\underbrace{(t_0 - \varepsilon, t_0 + \varepsilon) \cap X}_{\subset A}\big) \subset A$

So $[0, t_0] \subset A \Rightarrow t_0 \in A$

If $t_0 < 1$, then you can find some $s \in [t_0, t_0 + \varepsilon) \cap [0,1]$

& then $[0,s] \subset A$. So $s \in R$ yet $s > t_0 = \sup R$

So $t_0 = 1 \Rightarrow [0, t_0] = [0,1] \subset A$

So $A^c = \phi$.

Thm 2: $f : X \longrightarrow Y$ cts

X conn. $\Rightarrow f(X)$ conn.

pf: Assume $A \subset f(X)$ is clopen & not equal to $\emptyset$ or to $f(X)$

Then $f^{-1}(A)$ is clopen in X and different from $\emptyset, X$

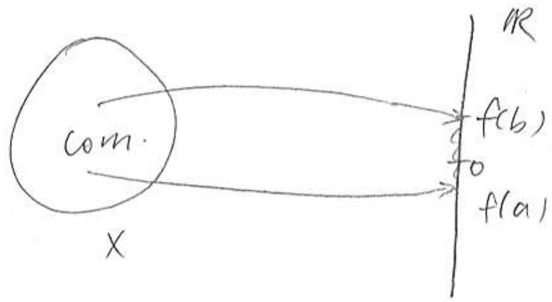
$(\Rightarrow \Leftarrow)$

Thm 3: (IVT)

$f: X \longrightarrow \mathbb{R}$ cts. X conn. $f(a) < 0 < f(b) \Rightarrow \exists x_0,\ f(x_0) = 0$

pf: Consider $f(X)$, it is conn as an image of a conn set

$0 > f(a) \in f(X)$



$0 < f(b) \in f(X)$

So $0 \in f(X)$

By thm 1. $\exists x_0 \in X$ s.t. $f(x_0) = 0$

Thm: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ cts. $\exists a, b \in \mathbb{R}^n$ s.t. $f(a) < 0 < f(b)$

$\Rightarrow \exists x_0 \in \mathbb{R},\ f(x_0) = 0$

Lem: If $X = \bigcup A_\alpha$ with 1. A_α conn

2. $\bigcap A_\alpha \neq \emptyset$

then X conn.

$\Rightarrow \mathbb{R}^n$ conn.