Sep 24, 2012 Mat267

Calculus (p)Review

Theorem

If $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is twice differentiable, then $\psi_{xy} = \psi_{yx}$

$$\frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y}$$

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y}$$

"Second Derivative Commute"

example

$$\psi = \frac{1}{x}\cos y$$

$$\psi_{xy} = \left(-\frac{1}{x^2}\cos y\right)_y = -\frac{1}{x^2}(-\sin y)$$

$$\psi_{yx} = \left(\frac{-\sin y}{x}\right)_x = -\frac{-\sin y}{x^2}$$

$\parallel$

Combinatorial Analog "∂x"

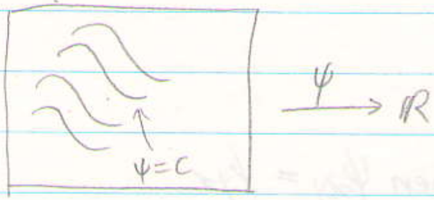
If $\psi : \mathbb{Z}^2 \longrightarrow \mathbb{R}$, define $(\delta_x \psi)(x,y) = \psi(x+1, y) - \psi(x,y)$ and $(\delta_y \psi)(x,y) = \psi(x, y+1) - \psi(x,y)$

Claim

$\delta_y \delta_x \psi = \delta_x \delta_y \psi$

proof

exercise



$(\nabla\psi)\cdot(y') = 0$

An ODE whose solution is $\psi(x,y) = c$ is $\psi_x + \psi_y y' = 0$

How can we tell if

$M + Ny' = 0 \quad \cdots (1)$

$M\,dx + N\,dy = 0$

Came from the procedure above?

$\rightarrow$ No chance unless $M_y = N_x \quad \cdots (2)$

If (2) holds, we say that (1) is exact

Theorem

If in some rectangle in the plane M, N & their first derivatives exist and are continuous then $\exists\psi$ on some rectangle s.t. $M = \psi_x$ and $N = \psi_y$ if and only if $M_y = N_x$.

proof

$(\Rightarrow)$ Already done, follows from $\psi_{xy} = \psi_{yx}$

$(\Leftarrow)$ Assume $M_y = N_x$

First find $\chi_x = M$ (possible, just integrate M with respect to x)

Want $\psi = \chi + \phi$, s.t. $\psi_y = N$

$\chi_x + \phi_y = N$ meaning

$\phi_y = \underbrace{N - \chi_y}_{\text{known}}$

must take ϕ with $\phi_x = 0$ meaning $\phi = \phi(y)$. ϕ depends only on y

Possible if RHS does not depend on x, meaning

$$\begin{aligned} 0 &= (N - \chi_y)_x \\ &= N_x - \chi_{yx} \\ &= N_x - \chi_{xy} \\ &= N_x - M_y \end{aligned}$$

$\square$

example

Solve $(\underbrace{y\cos x + 2xe^y}_{M}) + (\underbrace{\sin x + x^2 e^y + 2}_{N})y' = 0$

Exact?

$M_y = \cos x + 2xe^y$
$N_x = \cos x + 2xe^y$ $\Big\} \Rightarrow$ exact!

$\chi_x = y\cos x + 2xe^y$

$\chi = y\sin x + x^2 e^y$

Want $\psi = \chi + \phi(y)$

$N = \psi_y = y\sin x + x^2e^y + \phi_y(y)$

$\parallel$

$\sin x + x^2 e^y + 2$

$\Rightarrow \phi_y = 2$

$\Rightarrow \phi = 2y$

$\Rightarrow \boxed{\psi = y\sin x + x^2e^y + 2y = c}$

$M + Ny' = 0$

$\left(\frac{M}{\mu}\right) + \left(\frac{N}{\mu}\right)y' = 0$

How do I uncover the tracks and find μ?

Given $M + Ny' = 0$, find μ s.t. $\mu M + \mu N y' = 0$ is exact

$(\mu M)_y \overset{?}{=} (\mu N)_x$

$\mu_y M + \mu M_y \overset{?}{=} \mu_x N + \mu N_x$

$\mu_y M - \mu_x N = \mu (N_x - M_y)$

This is a PDE!

Assume μ depends only on x

$\Rightarrow -\mu_x N = \mu(N_x - M_y)$

$\underbrace{\frac{\mu_x}{\mu}}_{\text{depends only on } x} = \underbrace{\frac{M_y - N_x}{N}}_{\text{lucky if this depends only on } x}$