Nov. 6, 2012

Claim

If $\psi'(t) = A(t)\psi(t)$ then Ψ is either regular for all t or singular for all t

proof 1

Use existence & uniqueness

Debts

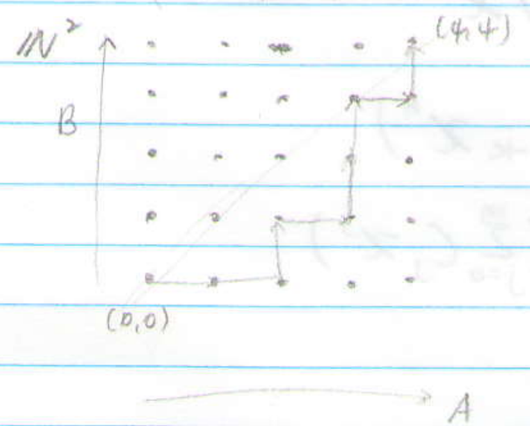
1. Make length of existence/uniqueness interval explicit
2. Do proof 2 using the "Wronskian" & det

Power series - an unusual motivation

1. Power series are keepers combinatorial info.
2. Recursion relation $\Leftrightarrow$ differential equations "polynomial coefficients"

$$A_n = \frac{1}{n+1}\binom{2n}{n} \xrightarrow[\text{of the sequence"}]{\text{"generating function}} F = \sum_{n=0}^{\infty} A_n x^n$$

$$C_n = \left(\begin{array}{l} \text{\# of below-diagonal} \\ \text{paths in the integer} \\ \text{lattice going from} \\ (0,0) \text{ to } (n,n) \end{array} \right) \rightsquigarrow G = \sum_{n=0}^{\infty} C_n x^n = 1 + x + 2x^2 + \cdots$$



00
10
20
21
31
32
33
43
44

$C_0 = 1$

$C_1 = 1$

$C_2 = 2$

$C_n =$ "nth Catalan number"

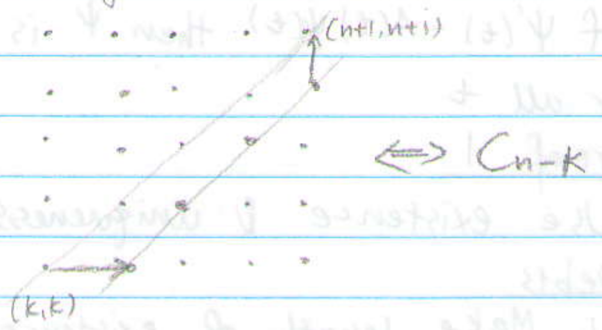
Game ends $(n+1, n+1)$. Let k be the maximal k s.t. (k,k) was a score in our game. $0 \le k \le n$

(C, G)

1. $C_0 = 1$

2. $C_{n+1} = \sum_{k=0}^{n} \underbrace{C_k}_{\text{ways to reach } (k,k)} \cdot C_{n-k}$

Going from (k,k) to $(n+1, n+1)$



$\Longleftrightarrow C_{n-k}$

$$C_2 = \sum_{k=0}^{1} C_k C_{1-k}$$
$$= C_0 C_1 + C_1 C_0$$
$$= 1 + 1$$
$$= 2$$

$$C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0$$
$$= 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1$$
$$= 5$$

$$C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0$$
$$= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1$$
$$= 14$$

Take (2) $\times x^{n+1} \cdot \sum_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} C_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} C_n C_{n-k} \right) x^{n+1}$$

$$\Rightarrow \quad G - 1 = x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} C_k C_{n-k} x^n \right)$$
$$= x \left(\sum_{i=0}^{\infty} C_i x^i \right) \left(\sum_{j=0}^{\infty} C_j x^j \right)$$
$$= x G^2$$

$G(0) = 1, \quad xG^2 - G + 1 = 0$

$$G = \frac{1 \pm \sqrt{1 - 4x}}{2x} \Rightarrow G = \frac{1 - \sqrt{1-4x}}{2x}$$

$$A_n = \frac{1}{n+1} \frac{(2n)!}{n!n!}$$

$$= \frac{(2n)!}{n!(n+1)!}$$

$$A_{n+1} = \frac{(2n+2)!}{(n+1)!(n+2)!}$$

$$= \frac{(2n+2)(2n+1)(2n)!}{(n+2)(n+1)!(n+1)!}$$

$$= \frac{2(2n+1)}{n+2} A_n$$

$(x^n)' = nx^{n-1}$

$x(x^n)' = nx^n$

$f(x) = \sum a_n x^n$

$xf' = \sum n a_n x^n$

$$\Rightarrow (n+2)A_{n+1} = (4n+2)A_n$$

$$\overset{\times x^{n+1}}{\underset{\sum_{n=0}^{\infty}}{\Rightarrow}} \sum_{n=0}^{\infty}(n+2)A_{n+1}x^{n+1} = \sum_{n=0}^{\infty}(4n+2)A_n x^{n+1}$$

$$\overset{n+1=m}{\Longrightarrow} -1 + \sum_{m=0}^{\infty}(m+1)A_m x^m = x\left(\sum_{n=0}^{\infty} 4nA_n x^n + \sum_{n=0}^{\infty} 2A_n x^n\right)$$

$$\Rightarrow \quad -1 + xF' + F = x(4xF' + 2F)$$

F satisfies,

1. $F(0) = 1$
2. $x(4x-1)F' + (2x-1)F + 1 = 0$ can be solved.

Alternatively by hard work.

G satisfies this equation $\Rightarrow G = F$

$\Rightarrow A_n = C_n$

$\Rightarrow$ (# histories of matches ending (n,n) with A always ending) $= \frac{1}{n+1}\binom{2n}{n}$

Challenge

1. Find a direct combinatorial proof of $C_n = A_n$
2. Compute $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$