

Nov. 6, 2012

Claim: if (3.3) was proved

If  $\Psi'(t) = A(t)\Psi(t)$ , then  $\Psi$  is either regular for all  $t$  or singular for all  $t$

proof ↗

Use existence &amp; uniqueness

Debts

1. Make length of existence/uniqueness interval explicit.
2. Do proof ↗ using the "Wronskian" &  $\det \begin{pmatrix} \Psi_1 & \Psi_2 \\ \Psi'_1 & \Psi'_2 \end{pmatrix}$

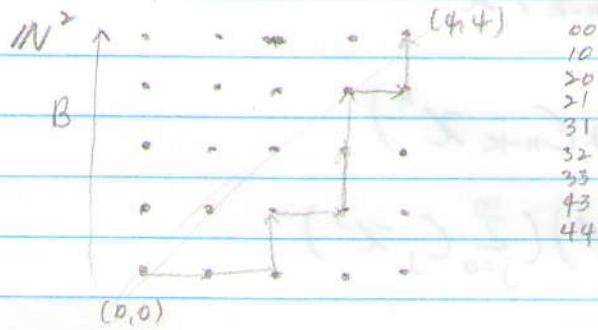
Power Series - an unusual motivation

1. Power series are keepers combinatorial info.
2. Recursion relation ↗ differential equations ↗ "polynomial coefficients"

$$A_n = \frac{1}{n+1} \binom{2n}{n} \quad \xrightarrow{\text{"generating function of the sequence" }} F = \sum_{n=0}^{\infty} A_n x^n$$

$$C_n = \left( \begin{array}{l} \# \text{ of below-diagonal} \\ \text{paths in the integer} \\ \text{lattice going from} \\ (0,0) \text{ to } (n,n) \end{array} \right) \rightsquigarrow G = \sum_{n=0}^{\infty} C_n x^n$$

$$= 1 + x + 2x^2 + \dots$$



$$C_0 = 1$$

$$C_1 = 1$$

$$C_2 = 2$$

$C_n$  = "nth Catalan number"

Game ends  $(n+1, n+1)$ . Let  $k$  be the maximal  $k$  s.t.  $(k, k)$  was a score in our game.  $0 \leq k \leq n$

(C, G)

1.  $C_0 = 1$  + No ref. ways to start at (0,0)  $\rightarrow$  (n+1, n+1)

Going from (k,k) to (n+1, n+1)

$$2. C_{n+1} = \sum_{k=0}^n C_k \cdot C_{n-k}$$

ways to reach (k,k)

$$C_2 = \sum_{k=0}^1 C_k C_{1-k}$$

$$= C_0 C_1 + C_1 C_0$$

$$= 1 + 1$$

$$= 2$$

$$C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0$$

$$= 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1$$

$$= 5$$

$$C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0$$

$$= 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1$$

$$= 14$$

Take (2)  $\times x^{n+1} \cdot \sum_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} C_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} C_k C_{n-k} \right) x^{n+1}$$

$$\Rightarrow G - 1 = x \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} C_k C_{n-k} x^n \right)$$

$$= x \left( \sum_{i=0}^{\infty} C_i x^i \right) \left( \sum_{j=0}^{\infty} C_j x^j \right)$$

$$= x G^2$$

$$G(0) = 1, x G^2 - G + 1 = 0$$

$$G = \frac{1 \pm \sqrt{1 - 4x}}{2x} \Rightarrow G = \frac{1 - \sqrt{1 - 4x}}{2x}$$

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$$A_n = \frac{1}{n+1} \frac{(2n)!}{n!n!}$$

$$= \frac{(2n)!}{n!(n+1)!}$$

$$A_{n+1} = \frac{(2n+2)!}{(n+1)!(n+2)!}$$

$$= \frac{(2n+2)(2n+1)(2n)!}{(n+2)(n+1)!(n+1)!}$$

$$= \frac{2(2n+1)}{n+2} A_n$$

$$\Rightarrow (n+2) A_{n+1} = (4n+2) A_n$$

$$\times x^{n+1} \Rightarrow \sum_{n=0}^{\infty} (n+2) A_{n+1} x^{n+1} = \sum_{n=0}^{\infty} (4n+2) A_n x^{n+1}$$

$$\stackrel{n+1=m}{\Rightarrow} -1 + \sum_{m=0}^{\infty} (m+1) A_m x^m = x \left( \sum_{n=0}^{\infty} 4n A_n x^n + \sum_{n=0}^{\infty} 2 A_n x^n \right)$$

$$\Rightarrow -1 + xF' + F = x(4xF' + 2F)$$

F satisfies,

$$1. \quad F(0) = 1$$

$$2. \quad x(4x-1)F' + (2x-1)F + 1 = 0 \quad \text{can be solved.}$$

Alternatively by hard work.

G satisfies this equation  $\Rightarrow G = F$ 

$$\Rightarrow A_n = C_n$$

$$\Rightarrow \begin{pmatrix} \# histories \\ \text{of matches} \\ \text{ending } (n,n) \\ \text{with } A \text{ always} \\ \text{ending} \end{pmatrix} = \frac{1}{n+1} \binom{2n}{n}$$

Faction

clock, draw

### Challenge:

1. Find a direct combinatorial proof of  $C_n = A_n$

2. Compute  $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$

$$x^N = \binom{N}{N}$$

$$x^N = ("x")^N$$

$$x^{N+1} = ("x")^{N+1}$$

$$x^{N+2} = "x"$$

$$x^{N+3} = "x"$$

$$x^{N+4} = "x"$$

$$x^{N+5} = "x"$$

$$x^{N+6} = "x"$$

$$x^{N+7} = "x"$$

$$x^{N+8} = "x"$$

$$x^{N+9} = "x"$$

$$x^{N+10} = "x"$$

$$x^{N+11} = "x"$$

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