Thm: (IFT)

$f: \mathbb{R}^n \circlearrowleft$ C^1 near $a \in \mathbb{R}^n$

$\exists\, Df(a)^{-1} \Rightarrow \exists$ nbhds $U \ni a$, $V \ni b = f(a)$ s.t

$\exists\, (f|_U)^{-1} : V \longrightarrow U$

if $f \in C^r$, then $(f|_U)^{-1} \in C^r$

WLOG, $Df(a) = I$, $a = b = 0$

Technical Lemma: f is Jelly-rigid near a

$\forall \varepsilon > 0$ $\exists$ nbhd $J_\varepsilon \ni a$ s.t. $\forall x, y \in J_\varepsilon$, ($J_\varepsilon = B(a, \delta_\varepsilon)$)

$\| f(y) - f(x) - (y-x) \| \leq \varepsilon \| y - x \|$

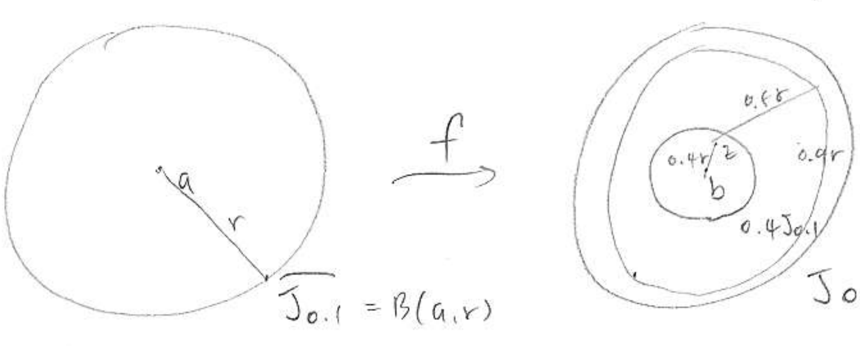
Part I: f is 1-1 on $J_{0.1}$

Indeed, if $f(x) = f(y)$ then $\| -(y-x) \| \leq 0.1 \| y - x \|$

$\Rightarrow \| y - x \| = 0$

$\Rightarrow y = x$

Part II: f is onto $0.4 J_{0.1}$



pf: Let $J_{0.1} = B(a,r)$

Show that every $z \in B(b, 0.4r)$ is in image(f).

Assume $z \notin$ image of f.

Consider $d: \overline{J_{0.1}} \longrightarrow \mathbb{R}_{\geq 0}$. $d(x) = \| f(x) - z \|$

d is cts on a cpt set, hence it attains it min

at some $x_0 \in \overline{J_{0.1}}$ as $z \notin im(f)$. $d(x_0) > 0$

Option 1: $x_0 \in$ int $\overline{J_{0.1}}$.

Consider $x_1 = x_0 + \delta(z - f(x_0))$ s.t.

1. $\delta < 1$

2. r is small enough s.t. $x_1 \in J_{0.1}$

e.g.: Show that $d(x_1) < d(x_0)$ $(\Rightarrow \Leftarrow)$

option 2: $x_0 \in$ bd $J_{0.1}$, but then $d(x_0) \geq 0.5r$

Yet $d(b) \leq 0.4r$ $(\Rightarrow \Leftarrow)$

Take $\underset{\substack{\cup\\ b}}{V} = 0.4 J_{0.1}$ and $\underset{\substack{\cup\\ a}}{U} = f^{-1}(V) \subset J_{0.1}$

and by part I & II, $f|_U$ is 1-1 and onto.

Part IV: $(f|_U)^{-1}$ is cts.

$$\|u - v\| \leq \varepsilon \|v\|$$
$$= \varepsilon \|u + (v-u)\|$$
$$\leq \varepsilon \|u\| + \varepsilon \|v - u\|$$
$$(1-\varepsilon)\|u - v\| \leq \varepsilon \|u\|$$
$$\|u - v\| \leq \frac{\varepsilon}{1-\varepsilon} \|u\|$$
$$\leq 0.5 \|u\|$$

$$\|v\| - \|u\| \le \|u - v\|$$
$$\le 0.5 \|u\|$$
$$\|v\| \le 1.5 \|u\|$$

$\|y - x\| \le 1.5 \|f(y) - f(x)\|$ at $x = f^{-1}(z_1)$ & $y = f^{-1}(z_2)$

This is $\|f^{-1}(z_1) - f^{-1}(z_2)\| \le 1.5 \|z_1 - z_2\|$

$\Rightarrow$ cts.