K-form on M: $w \cdot M \longrightarrow \bigcup_{p \in M} A^k(T_pM)$ s.t. $w(p) \in A^k(T_pM)$

on $\mathbb{R}^n$, $w = \sum_{I \in \binom{n}{k}} a_I(x)\psi_I = \sum_I a_I(x)\phi_{i_1} \wedge \phi_{i_2} \wedge \cdots \wedge \phi_{i_k}$

w is C^r means

$\forall I: a_I \in C^r \iff \forall C^r$ vector fields $Y_1, \dots, Y_k$,

$w(Y_1, \dots, Y_k)$ is C^r

Forms pullback!

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$. w on $\mathbb{R}^m$.

$\xi_1, \dots, \xi_k \in T_x\mathbb{R}^n$, $(f^*w)(\xi_1, \dots, \xi_k) = w(f_*\xi_1, \dots, f_*\xi_k)$

Def: $\Omega^k(\mathbb{R}^n)/\Omega^k(M)$ of all C^∞-forms on $\mathbb{R}^n/M$

$(w_1 + w_2)(\xi_1, \dots, \xi_k) = w_1(\xi_1, \dots, \xi_k) + w_2(\xi_1, \dots, \xi_k)$

$(\lambda w)(\xi_1, \dots, \xi_k) = \lambda w(\xi_1, \dots, \xi_k)$

$\wedge : A^k(V) \times A^l(V) \longrightarrow A^{k+l}(V)$

Def: $w \in \Omega^k(M)$. $\eta \in \Omega^l(M)$

$\underset{\Omega^{k+l}(M)}{\underset{\cap}{(w \wedge \eta)}}(\xi_1, \dots, \xi_k) = (w(x) \wedge \eta(x))(V_1, \dots, V_{k+l})$

$\xi_i = (x, V_i)$. $V_i \in T_xM$

Properties

1. Bilinear & associative
2. "super-symmetric"

   $w \wedge \eta = (-1)^{k \cdot l}\, \eta \wedge w$
3. $f^*(w \wedge \eta) = (f^*w) \wedge (f^*\eta)$

On $\mathbb{R}^3$

$\Omega^0(\mathbb{R}^3)$	$\Omega^1(\mathbb{R}^3)$	$\Omega^2(\mathbb{R}^3)$	$\Omega^3(\mathbb{R}^3)$
$\{f(x)\}$	$\left\{\begin{array}{l} a_1(x)\phi_1 \\ + a_2(x)\phi_2 \\ + a_3(x)\phi_3 \end{array}\right\}$	$\left\{\begin{array}{l} b_1(x)\phi_2\wedge\phi_3 \\ + b_2(x)\phi_3\wedge\phi_1 \\ + b_3(x)\phi_1\wedge\phi_2 \end{array}\right\}$	$\{c(x)\phi_1\wedge\phi_2\wedge\phi_3\}$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
fcts	$a(x) = \begin{pmatrix} a_1(x) \\ a_2(x) \\ a_3(x) \end{pmatrix}$	$b(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ b_3(x) \end{pmatrix}$	$c(x)$
	$\cong$	$\cong$	$\cong$
	V.F	V.F	functions

$$\left(\begin{array}{l} \underline{\text{0-form on } M} \\ w : M \longrightarrow \bigcup_{p\in M} \underbrace{A^0(T_pM)}_{\mathbb{R}} \\ w : M \longrightarrow \mathbb{R} \end{array} \right) \Longleftrightarrow \text{fct on } M$$

w is a 0-form and η is a k-form

$$\underbrace{(w \wedge \eta)}_{k\text{-form}}(\xi_1, \dots, \xi_k) = w(x)\cdot\eta(x)(V_1, \dots, V_k)$$

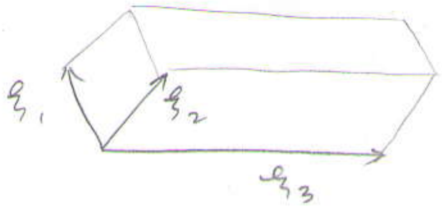
$$d : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$$

"Right def" (but too hard):

Sps $w \in \Omega^k(\mathbb{R}^n)$.

$$(dw)(\xi_1, \dots, \xi_{k+1}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{k+1}} \underbrace{w(\partial P(\varepsilon\xi_1, \dots, \varepsilon\xi_{k+1}))}_{\text{sum with signs of evals of } w \text{ on } k \text{ vectors.}}$$

$k = 2$



$= P(\xi_1, \cdots, \xi_3)$

$2(k+1)$ faces

∂P = Union of $2(k+1)$ faces

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Following def. let $f \in \Omega^0(\mathbb{R}^n)$ ~ f is a fct

want is

$$\underset{\Omega^1(\mathbb{R}^n)}{\underset{\cap}{(df)}}(\xi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(f(x + \varepsilon v) - f(x) \right) = D_\xi f$$

$\xi = (x, v)$