Thm: $X \subset \mathbb{R}^n$ cpt iff it is closed and bdd.

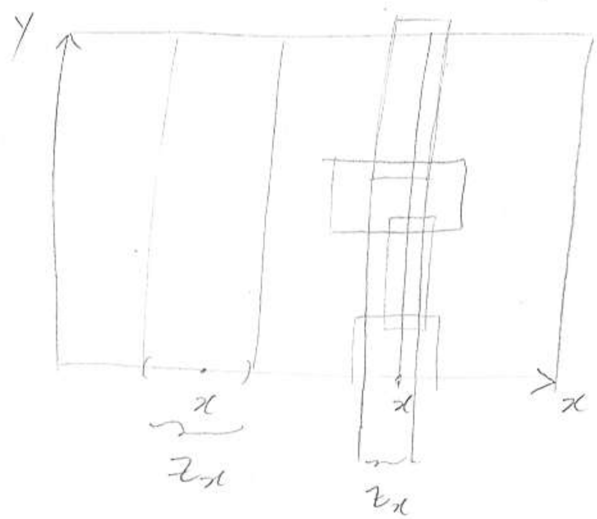
pf: $(\Rightarrow)$ done.

$(\Leftarrow)$

Lem: (Part of "Tychonoff's Thm")

If X & Y cpt. so is $X \times Y$

Lem²: If $W_\alpha = U_\alpha \times V_\alpha$ an open cover of $X \times Y$, then every $x \in X$ has an open nbhd Z_x s.t. $Z_x \times Y$ can be covered by fin. many W_α's



Lem² → Lem: The Z_x's cover X so by cptness. finitely many of the Z_x's cover x.

Call them $Z_{x_1}, \cdots, Z_{x_n}$

Now $Z_{x_i} \times Y$ cover $X \times Y$ and such is covered by fin. many W_α's

Take all W_α's used for all $Z_{x_i} \times Y$'s those cover $X \times Y$ and we've used just fin. many W_α's.

$\Rightarrow [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \times \cdots \times [a_n, b_n]$

"closed rectangles in $\mathbb{R}^n$ are cpt"

Lem 2: A closed subset A of a cpt X is cpt

If A is closed & bdd. then it is a closed subset of some retangle $[-M, M]^n$. hence A is a closed subset of cpt. hence it is cpt.

pf: If some U_α open in X cover A.

$A \subset \bigcup U_\alpha$. then $\{X \setminus A\} \cup \{U_\alpha\}$ covers X

hence some finite subcollection thereof covers X.

if this subcollection includes $X \setminus A$. throw it away and what remains still covers A.

Thm: If $f: X \longrightarrow Y$ cts & X cpt. then $f(X)$ also cpt.

pf: if $f(X) \subset \bigcup U_\alpha$ where each V_α open in Y

then $X \subset \bigcup_\alpha f^{-1}(V_\alpha)$ open cover. of X by $f^{-1}(V_\alpha)$'s which are open by cts of f by cptnesss of X. find $\alpha_1, \ldots, \alpha_n$ is $\bigcup_i f^{-1}(V_{\alpha_i})$ covers X and then $\{V_{\alpha_i}\}$ cover $f(X)$

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Cor: (Maximal Value Thm)

A cts fct $f : X \longrightarrow \mathbb{R}$ where X cpt is bdd.
and it attains its bounds

$\exists x_0 \in X$ s.t. $f(y) \le f(x_0)$ for every $y \in X$.

pf: By the previous thm. $f(X)$ cpt

Hence $f(X)$ is bdd. so f bdd and $f(X)$ closed

So it contains its limit pts.

$\sup f(X) \in f(X) \Rightarrow \sup_x f(x) = f(x_0)$ for some x_0 $\square$

Def: A sp X is called "connected" if there are no clopen sets in X except ϕ, X.

clopen ↓ open. & closed

e.g: $(0,1) \cup (2,3) \subset \mathbb{R}$

Not connected because $(0,1)$ & $(2,3)$ is clopen in X

Thm 1: $A \subset \mathbb{R}$ is connected iff it is generalized interval
(open or closed. finite or infinite)

Thm 2: X, Y conn. $\Rightarrow$ $X \times Y$ conn.

Thm 3: X conn & $f : X \longrightarrow Y$ cts $\Rightarrow$ $f(X)$ conn.

Cor: (Intermediate Value Thm)