Def. $Ext(A) = int(A^c)$

Def. $Bd\,A$ (← boundary) $= X \setminus (int(A) \cup ext(A))$

claim: i) $ext(A) = X \setminus \bar{A}$

ii) $int\,A = X \setminus \overline{A^c}$

iii) $Bd\,A = \bar{A} \cap \overline{A^c}$

Sep 26.

Def. A space $X$ is called "compact" if whenever you cover $X$ with open sets, finitely many of these already cover $X$.

$(X = \bigcup_{\alpha \in I} U_\alpha,\ U_\alpha \text{ open}) \Rightarrow (\exists F \subset I \text{ finite s.t. } X = \bigcup_{\alpha \in F} U_\alpha)$.

e.g. $\mathbb{R}$ is not compact. indeed, $X = \bigcup_{n>0} (-n, n)$ yet no finite subset.

e.g. A finite set is compact.

Thm: $[0,1]$ is compact ; $(0,1]$ is not compact.

*. Indeed. $(0,1] = \bigcup_n (\frac{1}{n}, 1]$ but cannot find a finite number of subset.

Thm. A cont. function on a compact space is bounded.

* bdd: $\exists M$ s.t. $\forall x \in X$ $|f(x)| < M$.

Def. $A \subset X$ is compact if whenever you cover $A$ with open sets, you can find a finite subcover naming, finitely many of those open sets already cover $A$.

Claim: The two interpretations are the same.

proof: key pt: $V \subset A$ is open iff $\exists U \subset X$ s.t. $U \cap A = V$

proof (i: $[0,1]$ is compact) Let $U_\alpha$ be open set s.t. $\bigcup U_\alpha = I = [0,1]$

Let $G = \{y \in [0,1] : \text{the interval } [0,y] \text{ can be contained by finitely many of the } U_\alpha\text{'s}\}$

then $0 \in G$ because $[0,0] = \{0\} \in I$. $\exists \alpha_0$ s.t. $0 \in U_{\alpha_0}$ and then $[0,0] \subset U_{\alpha_0}$.

So $[0,0]$ has a finite subcover.

$G$ is bdd & nonempty. so $g = \sup G$. exists, $0 \le g \le 1$

Claim1 $g \in G$.

proof: as $g \in \bigcup U_\alpha$, can find some $\alpha_0$ s.t. $g \in U_{\alpha_0}$. $U_{\alpha_0}$ is open. so $\exists\ \varepsilon > 0$ s.t. $(g-\varepsilon, g+\varepsilon) \subset U_\alpha$.

As $g = \sup G$, $\exists y \in G$ s.t. $g - \varepsilon < y \le g$ but then by def. of $G$, $\exists$ finitely many $U_{\alpha_1} \dots U_{\alpha_n}$ s.t. $[0,y] \subset \bigcup_{i=1}^{n} U_{\alpha_i}$, and then $\bigcup_{i=0}^{n} U_{\alpha_i} \supset (g-\varepsilon, g+\varepsilon) \cup [0,y] \supset [0,g]$

So $[0,g]$ has a finite cover by $U_\alpha$'s so $g \in G$.

Claim2 $g > 0$

proof: If some $U_{\alpha_0}$ covers 0, it also covers $[0,\varepsilon)$ for some $\varepsilon>0$. $\Rightarrow \sup G \geq \varepsilon > 0$.

Claim3: $g=0$ (and then we are done)

proof: If not, let's suppose $g<1$. then some $U_{\alpha_0}$ covers $g$, some $U_{\alpha_1}, \dots U_{\alpha_n}$ cover $[0,g]$,
since $U_{\alpha_i}$ is open, it covers $(g-\varepsilon, g+\varepsilon)$ for some $\varepsilon$ so $\bigcup^n U_{\alpha_i} \supset [0,g+\varepsilon) \supset [0, g+\frac{\varepsilon}{2}]$
there is a contradict.

Thm: A subset $A \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ A is closed and bdd.

proof: ($\Rightarrow$) $A \subset \bigcup_n U(0,n) = \mathbb{R}^n$

A is ~~com~~compact $\Rightarrow \exists N$ s.t. $A \subset \bigcup_{n=1}^{N} U(0,n) = U(0,N)$ (all is cont.) $\Rightarrow$ A is bdd. ▨.

Sep 28.

* Compact: $A \subseteq X$ "compact"

$(A = \bigcup_{\alpha \in I} U_\alpha,\ U_\alpha \text{ open in } A) \Rightarrow (\exists F \subset I \text{ finit. s.t. } A = \bigcup_{\alpha \in F} U_\alpha)$

$\Leftrightarrow (A \subset \bigcup_{\alpha \in I} V_\alpha,\ V_\alpha \text{ open in } X) \Rightarrow (\exists F \subset I \text{ finit, s.t. } A \subset \bigcup_{\alpha \in F} V_\alpha)$.

* Bdd: $\exists M\ \forall x \in X\ \|x\| < M$ or $\exists N\ \forall x \in X\ |x| < N$.

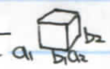
proof: ($\Rightarrow$) (continue). A is bdd is already proved.

Closed: Let $a \notin A$. for any $n$, consider $D_n = \{b : d(a,b) > \frac{1}{n}\}$ easy to show $D_n$ open.

$\bigcup_{n=1}^{\infty} D_n = \mathbb{R}^n \setminus \{a\} \supset A$. so by compactness, $\exists N$ s.t. $D_N = \bigcup_{n=1}^{N} D_n \supset A$

~~$D_N = \bigcup_{n=1}^{N} D_n \supset A$~~, so $A \subset D_N$, So $\mathbb{R}^n \setminus A \supset \mathbb{R}^n \setminus D_N \supset U(A, \frac{1}{N})$

So $A^c$ is open and A is closed.

* $\longleftarrow$  is compact $[a_1,b_1] \times [a_2,b_2] \times [a_3,b_3]$

*Q: $\longleftarrow$ if $X$ & $Y$ are compact, is $X \times Y$ compact?

Def.: Suppose $(X, d_1)$ & $(Y, d_2)$ are metric define on $X \times Y$ $d((x,y),(x',y')) = d_1(x,x') + d_2(y,y')$ ①
or $\sqrt{d_1(x,x')^2 + d_2(y,y')^2}$ ② or $\max(d_1(x,x'), d_2(y,y'))$ ($\infty$)

Claim: All three possibilities define a metric, and open sets with any of the options are the same.

open balls: 

$\max(d_1(x,x'), d_2(y,y')) < \varepsilon$

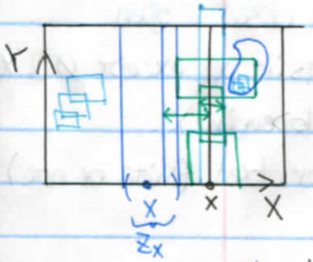
* $U_{d_\infty}((x,y),\varepsilon) = \{(x,y) : d_\infty((x,y),(x',y')) < \varepsilon\} = \{(x',y') : d_1(x,x') < \varepsilon,\ d_2(y,y') < \varepsilon\}$
"an open square around $(x,y)$"

Thm: If $X$ & $Y$ are compact so is $X \times Y$

proof: Let $\{W_\alpha\}$ be an open cover of $X \times Y$.

Lemma: WLOG (without loss of generality) each $W_\alpha$ is of the form
$W_\alpha = U_\alpha \times V_\alpha$ where $U_\alpha$ is open in $X$ and $V_\alpha$ is open in $Y$.

Indeed, each $W_\alpha$ is a union of symbols. so consider the cover of $X\times Y$ by all these squares. If I find a finite subcover using these squares. it clearly defines a finite subcover using the original $W_\alpha$'s.




Claim: If $X\times Y$ is covered by $U_\alpha\times V_\alpha$, then for every $x_0\in X$ we can find $\varepsilon>0$ s.t. $U\,\cancel{(x,y)}\,(x_0,\varepsilon)\times Y$ is covered by finitely many of the $U_\alpha\times V_\alpha$'s.

proof: By compactness of $Y$, $\exists$ $F$ finite s.t. $\bigcup_{\alpha\in F} \cancel{\forall} U_\alpha\times V_\alpha \supset \{x_0\}\times Y$.

But $\bigcup_{\alpha\in F} U_\alpha\times V_\alpha \supset \bigcup_{\alpha\in F}\left(\bigcap_{\beta\in F} U_\beta\right)\times V_\alpha \supset \underbrace{\left(\bigcap_{\beta\in F} U_\beta\right)\times Y}_{\text{open}}$

Sep 30.

* Why is claim ⇒ lemma above?

The $Z_x$'s cover $X$, by compactness, finitely many of the $Z_x$'s cover $X$.

Call them $Z_{x_1}, \dots, Z_{x_n}$.

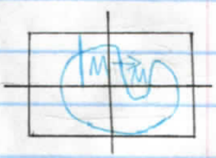
But now $Z_{x_i}\times Y$ cover $X\times Y$ and each is covered by finitely $W_\alpha$'s

Take all $W_\alpha$'s used for all $Z_{x_i}\times Y$'s. Those cover $X\times Y$ and we've used finitely many $W_\alpha$'s.

$\Rightarrow \underbrace{\underbrace{[a_1,b_1]}_{\text{compact}}\times\underbrace{[a_2,b_2]}_{\text{compact}}}_{\text{compact}}\times[a_3,b_3]\times\dots\times[a_n,b_n]$ compact (by induction).

i.e. closed rectangles in $\mathbb{R}^n$ are compact.

* Lemma 2: a subset $A$ (closed) of a compact $X$ is compact.




Why Lemma 2 ⇒ theorem above?

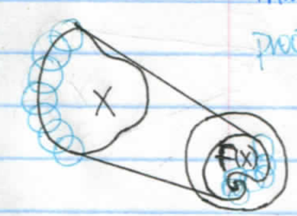
If $A$ is closed & bdd, then it is closed subset of some rectangle $[-M,M]^n$ hence $A$ is a closed subset of empty hence it is closed. □

* proof (of lemma 2): If some $U_\alpha$ open in $X$ cover $A$, $A\subset\bigcup U_\alpha$, then $\{X\setminus A\}\cup\{U_\alpha\}$ covers $X$ hence some finite ~~so~~ sub collection there of covers $X$.

If this subcollection includes $X-A$, throw it away! And what remains still cover $A$. □


taste good?

Thm. If $f: X\to Y$ is cont. and $X$ is compact, then $f(X)$ is also compact.




proof. If $f(X)\subset\bigcup V_\alpha$ where each $V_\alpha$ is open in $Y$, then $X\subset\bigcup f^{-1}(V_\alpha)$ is an open cover of $X$ by $f^{-1}(V_\alpha)$'s, which are open by continuity of $f$.

By compactness of $X$, find $\alpha_1,\dots,\alpha_n$ s.t. $\bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$ cover $X$, and then $\{V_{\alpha_i}\}$ cover $f(X)$.

Corollary. (Maximal value thm) A cont. function $f: X\to\mathbb{R}$ where $X$ is compact is bdd, and it attains its bounds; $\exists\, x_0\in X$ s.t. $f(y)\le f(x_0)$ for every $y\in X$.

proof: By the previous thm, $f(X)$ is compact, hence $f(X)$ is bdd so $f$ is bdd and $f(X)$ is closed, so it contains its limit pts. so $\sup f(X) \in f(X) \Rightarrow \sup_x f(x) = f(x_0)$ for some $x_0$. $\square$

Sets both closed & open.

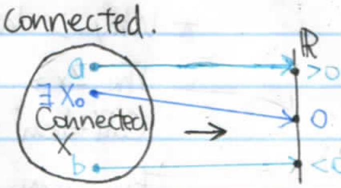
Def. A space $X$ is called "connected" if there are no clopen sets in $X$ except $\emptyset$ & $X$.

e.g. $X=(0,1) \cup (2,3) \subset \mathbb{R}$ is not connected because $(0,1)$ & $(2,3)$ is clopen in $X$.

Thm. $A \subset \mathbb{R}$ is connected iff it is a generalised interival (open or closed, finite or not)

Thm. $X, Y$ connected $\Rightarrow X \times Y$ is connected.

Thm. If $X$ is connected & $f: X \to Y$ is cont. then $f(X)$ is connected.




Cor. The intermediate value thm.

(TBC next week)