Thm: (Inverse Fct Thm)

If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is ctsly diff (C^1) near $a \in \mathbb{R}^n$ &

$Df(a)^{-1}$ exists, then there is a nbhd U of a & V of $b = f(a)$

s.t. $f|_U : U \longrightarrow V$ is invertible

Furthermore, if f is C^r, then so is $(f|_U)^{-1}$

Comment: WLOG, $Df(a) = I$.

$a = 0 = b$

Technical lemma: f is Jelly-Rigid near a

$$f(x) - f(y) \sim y - x$$

$\forall_{\varepsilon > 0}$ $\exists_a$ nbhd $J = J_\varepsilon = B(0, \delta)$ s.t.

$\forall_{x, y \in J}$, $\| f(y) - f(x) - (y - x) \| \leq \varepsilon \| y - x \|$

wrong pf: $y = x + (\underbrace{y - x}_{h})$.

$$f(y) = f(x + (y - x))$$

$$= f(x) + Df(x)(y - x) + \varphi(y - x) \quad \text{"tiny"}$$

$$= f(x) + (I + \underbrace{B}_{\text{small matrix}})(y - x) + \varphi(y - x)$$

$$\Rightarrow \| f(y) - f(x) - (y - x) \| \leq \underbrace{\| B(y - x) \|}_{\ll \varepsilon \| y - x \|} + \underbrace{\| \varphi(y - x) \|}_{\ll \varepsilon \| y - x \|}$$

Aside: MVT in $\mathbb{R}$, $\dfrac{f(b) - f(a)}{b - a} = f'(c)$ $c \in (a, b)$

MVT in $\mathbb{R}^n$, if $f : \mathbb{R}^n \to \mathbb{R}$ is diff on the line l between a & b then $\exists\, c \in l$ s.t

$$\underbrace{f(b) - f(a)}_{\in \mathbb{R}} = \underbrace{Df(c)}_{1\times n} \cdot \underbrace{(b-a)}_{n\times 1}$$

pf of MVT in $\mathbb{R}^n$: Consider $g : [0,1] \to \mathbb{R}$

$$g(t) = f(a + t(b-a))$$

By 1D-MVT: $\exists\, t_0 \in [0,1]$,

$$g(1) - g(0) = g'(t_0)\cdot(1-0)$$

$$f(b) - f(a) = Df(c)\cdot(b-a)$$

$a + t_0(b-a)$

pf: Find $c_1, \dots, c_n$ between x and y s.t.

$$f_i(y) - f_i(x) = Df(c_i)_i \cdot (y-x)$$

"ith row"

$$= (I + D_i)_i\,(y-x)$$

D_i has all entries less than ε/n

$$= y_i - x_i + d_i(y-x)$$

small

$$\Rightarrow \|f_i(y) - f_i(x) - (y_i - x_i)\| = \|d_i(y-x)\|$$

$$\le n\|d_i\|\|y-x\|$$

on all of J. provided J is small enough

$$\Rightarrow \|f(y) - f(x) - (y-x)\| \le \varepsilon\|y-x\|$$