$\phi' = f(x, \phi)$, $\phi(x_0) = y_0$

f is uniformly Lipschitz : $|f(x, y_1) - f(x, y_2)| < K|y_1 - y_2|$

$\Rightarrow$ solution exists and is unique

proof

equation $\iff \phi(x) = y_0 + \int_{x_0}^{x} f(x, \phi(x))\,dx$

$\phi_0 \equiv y_0$; $\phi_n(x) = y_0 + \int_{x_0}^{x} f(x, \phi_{n-1}(x))\,dx$

$\phi_n \xrightarrow{\text{uniformly}} \phi$; ϕ is a solution

Uniqueness ?

Suppose ϕ & ψ both solutions

$|\phi(x) - \psi(x)| = |\int_{x_0}^{x} (f(x, \phi(x)) - f(x, \psi(x)))\,dx|$

$\leq \int_{x_0}^{x} K \cdot |\phi(x) - \psi(x)|\,dx$

$\chi(x) = |\phi(x) - \psi(x)|$ $\chi(x) \geq 0$

$\chi(x) \leq K \cdot \int_{x_0}^{x} \chi(x)\,dx$

$u(x) = e^{-Kx} \int_{x_0}^{x} \chi(x)\,dx$, $x \geq x_0$. $u(x) \geq 0$

$u'(x) = -Ke^{-Kx} \int_{x_0}^{x} \chi\,dx + e^{-Kx} \chi(x)$

$= e^{-Kx} \left(\chi - K \int_{x_0}^{x} \chi\,dx \right)$

≤ 0

$u(x_0) = 0$

$\Rightarrow u \leq 0$

$\Rightarrow u \equiv 0$

$\Rightarrow \int_{x_0}^{x} \chi \equiv 0 \Rightarrow \chi \equiv 0 \Rightarrow \phi \equiv \psi$ $\square$

Systems of ODE's

$\phi_1' = f_1(x, \phi_1, \phi_2) \quad \phi_1(x_0) = y_{01}$

$\phi_2' = f_2(x, \phi_1, \phi_2) \quad \phi_2(x_0) = y_{02}$

$\phi : \mathbb{R} \to \mathbb{R}^n$

$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}, \quad \phi(x_0) = y_0 = \begin{pmatrix} y_{01} \\ \vdots \\ y_{0n} \end{pmatrix}$

$\phi' = f(x, \phi)$

$f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Theorem

On $R = [x_0 - a, x_0 + a] \times \{y : |y - y_0| \le b\}$

$$|y - y_0| = \left(\sum (y_i - y_{0i})^2\right)^{\frac{1}{2}}$$

f is uniformly Lipschitz

Set $\delta = \min(a, \frac{b}{M})$ M a bound on f on $\mathbb{R}$

$\Rightarrow \exists!$ a solution to our system $(*)$ on $[x_0 - \delta, x_0 + \delta]$

proof

Same proof except $|\,-\,| \Rightarrow |\,\cdot\cdot\,|$

absolute values $\quad$ euclidean distance

Higher Order ODE's

$y^{(3)} = f(x, y, y', y''), \quad y(x_0) = y_0, \; y'(x_0) = y_1, \; y''(x_0) = y_2$

$\Updownarrow \; y = \phi_0, \; y' = \phi_1, \; y'' = \phi_2$

$\phi_0' = \phi_1$

$\phi_1' = \phi_2$

$\phi_2' = f(x, \phi_0, \phi_1, \phi_2), \quad \phi_0(x_0) = y_0, \; \phi_1(x_0) = y_1, \; \phi_2(x_0) = y_2$

$\phi = F(x, \phi), \quad F(x, v_0, v_1, v_2) = \begin{pmatrix} v_1 \\ v_2 \\ f(x, v_0, v_1, v_2) \end{pmatrix}$

If $f(x, v_0, v_1, \cdots, v_n)$ is uniformly Lipschitz in the v's then $y^{(n+1)} = f(x, y, y', \cdots, y^{(n)})$ with appropriate initial condition has an unique solution.

diff. calculus $\Rightarrow$ "calculus of variations"

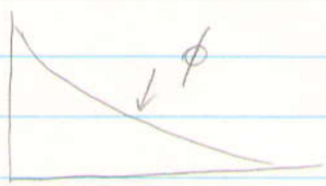
"diff. calculus in ∞-dimensions"

"minimize a functional"

functional: $\left\{ \begin{matrix} \text{space of all} \\ \text{functions} \end{matrix} \right\} \xrightarrow{J} \mathbb{R}$

$J : \{ \sim \} \longrightarrow \mathbb{R}$

Brachistochrone



$\xrightarrow{J}$ $J(\phi)$ = time it takes to slide

min J ..... "J'"

$\boxed{J' = 0}$

$\Updownarrow$

ODE 2nd order