$0, 1, +, \cdot$

Are matrices a field?

No, though almost.

1. $\times$ (multiplication) is not always defined.

2. many matrices other than 0, have no inverse

3 in general: $A \cdot B \neq B \cdot A$ (sometimes $AB = BA$) ie $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ 0 & \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$A \cdot B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$

$A \cdot B \neq BA$

$B \cdot A = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

Tutorial Any linear transformation

$T: V \longrightarrow W$ can be represented as a matrix

if $\{e_1, \dots, e_n\}$ - basis of V

$\{f_1, \dots, f_m\}$ - basis of W $[T]_e^f$

$T(e_j) = \sum_{i=1}^{m} a_{ij} f_i \rightarrow$ we write $T = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{pmatrix} \rightarrow \in M_{m \times n}$

① Write T in a matrix form

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $\quad e = (1,0)(0,1)$ / $f = (1,0,0), (0,1,0), (0,0,1)$

$T(a_1, a_2) = (2a_1 - a_2,\ 3a_1 + 4a_2,\ a_1)$

$T(1,0) = (2,3,1)$

$T(0,1) = (-1,4,0)$

$$T = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$$

② ——————

$T: M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ in stand basis

$$e = \left\{ \underset{e_1}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{e_3}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{e_4}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$$

$$f = \underset{f_1, f_2, f_3}{\{1, x, x^2\}}$$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + b x^2$$

Sol'n:

$T(e_1) = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 = f_1$

$T(e_2) = T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1 + x^2 = f_1 + f_2$

$T(e_3) = T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 = e_3 \in \ker T$

$T(e_4) = T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 2x = 2f_2$

$$T \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

if we multiplying this matrix with e_3 $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ we get 0 matrix

3) Let $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$

which is given by $T(A) = A^t$

What is the matrix repr. of T? (in stand. basis?)

<u>Soln:</u> If $A = (a_{ij})$, then $A^t = (a_{ij})$

$T(e_1) = e_1$, $T(e_2) = e_3$, $T(e_3) = e_2$ $\quad T(e_4) = e_4$

Then

$$T \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\dim = n^2$

In general, $T: M_{n\times n}(\mathbb{R}) \circlearrowleft$, $T(A) = A^T$, basis E_{ij}

symmetric to diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & & & \end{pmatrix}$$

make a symmetric

$$T(T(E_{ij})) = T(E_{ji}) = E_{ij}$$

④ Define $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$$

what is the matrix repr'n of T? (in stand. basis)

So

Sol'n: Apply T to the basis el's.

$$e = \{1, x, x^2\} \qquad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$e_1 \quad e_2 \quad e_3$$

$$T(e_1) = T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0f_1 + 2f_2 + 0f_3 + 0f_4$$

$$T(e_2) = T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1f_1 + 2f_2 + 0f_3 + 0f_4$$

$$T(e_3) = T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = 0f_1 + 2f_2 + 0f_3 + 2f_4$$

$$\Rightarrow T \sim \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{Ex: } \operatorname{Ker}(T) \overset{?}{\neq} \{0\}$$

5) $T: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$

$$T(A) := tr(A) = a_{11} + a_{22}$$

$$\left.\begin{aligned} T(f_1) &= 1 \\ T(f_2) &= 0 \\ T(f_3) &= 0 \\ T(f_4) &= 1 \end{aligned}\right\} \quad T \sim \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

⑥ Prove that a projection

$T: V \longrightarrow V$

can be represented by a diag. matrix

proof: we have to find a basis $e = \{e_1, \dots, e_n\}$ of V s.t. $[T]_e$ = diagonal matrix.

Let $T: V \to W \subset V$ be a procetion on $W \subset V$ $(\dim W = m < n)$

i.e. $V = W \oplus W_1$, $T(x+y) = x$, $x \in W$, $y \in W_1$

Obj: to show that $\exists$ basis e in V s.t. T repr't in this basis by a diag. matrix.

Construction of the basis with this property!

First choose a basis of W, $\{e_1, \dots, e_m\}$;

Then $\exists$ a complement of this basis to a basis of V, $\{e_1, \dots, e_m, e_{m+1}, \dots e_n\}$

Let us calculate the matrix $[T]_e$

In this basis T looks like

$$\underbrace{T(a_1e_1 + \dots + a_me_m)}_{x \in W} + \underbrace{(a_{m+1}e_{m+1} + \dots + a_ne_n)}_{y \in W_1} = a_1e_1 + \dots + a_me_m = x$$

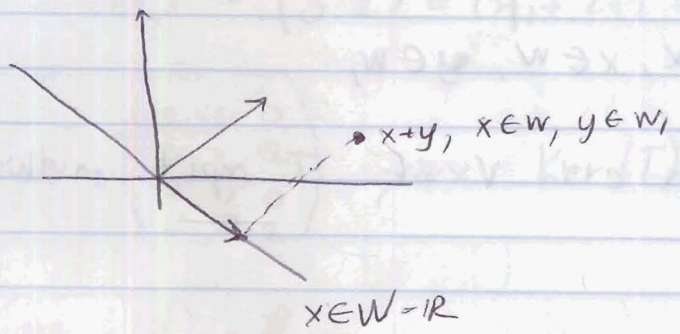
So : $T(e_1) = e_1$ $\qquad$ $T(e_i) = 0$

$\vdots \qquad \vdots$

$T(e_m) = e_m$ $\qquad$ $m < i \leq n$

Then the matrix $[T]_e$

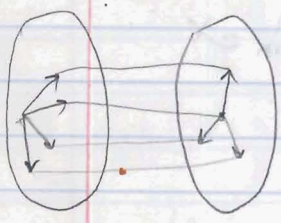
$$T \sim \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ \vdots & \vdots & 0 & 1 & & \vdots \\ 0 & 0 & 0 & 0 & & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_m \\ a_{m+1} \\ \vdots \\ a_n \end{pmatrix} \overset{?}{=} \begin{pmatrix} a_1 \\ \vdots \\ a_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$



$\dim V = \dim W$

7) Show If $T: V \to W$ is a l.t then $\exists$ a basis e, f of V s.t. $[T]_e^f$ is a diagonal matrix

proof: Let us take any basis $e = \{e_1, \ldots, e_n\}$ in V. Consider $\{T(e_1), \ldots, T(e_n)\}$. Then these vectors generate the range $\mathbb{R}(T)$ of T. Let us choose a basis of $\mathbb{R}(T)$ among these vectors.



Let $\{T(e_{i_1}), \ldots, T(e_k)\}$ $k \leq n$ be a basis of the range $R(T)$

Let us rearrange the vectors $\{e_1, \ldots, e_n\}$ in such a way that

they are : $\{e_{i_1}, \ldots, e_{i_k}, e_{i_{k+1}}, \ldots, e_{i_n}\} \left(\xrightarrow{T} \{e_{i_1}, \ldots, e_{i_k}\} \right)$

Extend the basis $\{T(e_{i_1}), \ldots, T(e_{i_k})\}$ of $R(T)$ to a basis of W.

$$\left\{ \underset{\substack{\| \\ f_1}}{T(e_{i_1})} \ldots \underset{\substack{\| \\ f_k}}{T(e_{i_k})}, f_{k+1}, \ldots, f_k \right\}$$

Assertion: $[T]_e^f$ is diagonal. $= \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & 1 & \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \end{pmatrix}$