Thm: $\exists!$ linear operator $d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ s.t.

1. If $f \in \Omega^0(\mathbb{R}^n)$ $df = \sum \frac{\partial f}{\partial x_i} \cdot dx_i$ 2. $\omega \in \Omega^k, \eta \in \Omega^l$, $d(\omega \wedge \eta) = (d\omega)\wedge\eta + (-1)^k \omega \wedge (d\eta)$
3. $d^2 = 0$: $d \circ d = 0$ if $\omega \in \Omega^k(\mathbb{R}^n)$ then $d(d\omega) = 0$ in $\Omega^{k+2}(\mathbb{R}^n)$

$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$
$d(dx_I) = d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = 0.$

$$d\omega = d\left(\sum_{I \in \binom{n}{k}} a_I(x) \cdot dx_I\right) = \sum_I d(a_I dx_I) = \sum_I (da_I) \wedge dx_I \pm \sum a_I \wedge d(dx_I)$$
$$= \sum_I \left(\sum_i \frac{\partial a_I}{\partial x_i} dx_i\right) \wedge dx_I = \sum_{i,I} \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I.$$

$\Rightarrow$ we have prove uniqueness in thm!

$$d(b_1 dx_2 \wedge dx_3 + \cdots) = (db_1) \wedge dx_2 \wedge dx_3 + \cdots = \left(\frac{\partial b_1}{\partial x_1} dx_1 + \frac{\partial b_1}{\partial x_2} dx_2 + \frac{\partial b_3}{\partial x_3} dx_3\right) dx_2 \wedge dx_3 + \cdots$$
$$= \frac{\partial b_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial b_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \cdots$$
$$= \left(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}\right) dx_{123}.$$

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3) \xrightarrow{d} \Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

$$\{f(x)\} \quad \left\{\begin{array}{l} a_1(x)\,dx_1 \\ +a_2(x)\,dx_2 \\ +a_3(x)\,dx_3 \end{array}\right\} \longrightarrow \left\{\begin{array}{l} b_1(x)\,dx_2 \wedge dx_3 \\ +b_2(x)\,dx_3 \wedge dx_1 \\ +b_3(x)\,dx_1 \wedge dx_2 \end{array}\right\} \quad \{c(x)\,dx_1 \wedge dx_2 \wedge dx_3\}$$

functions $\xrightarrow[\nabla f]{\text{grad}(f)}$ vector fields $\xrightarrow[\nabla \times]{\text{curl}}$ vector fields. $\xrightarrow[\nabla \cdot B]{\text{div} B}$ functions

$$f \longrightarrow \begin{array}{l} (\partial f/\partial x_1)\,dx_1 \\ +(\partial f/\partial x_2)\,dx_2 \\ +(\partial f/\partial x_3)\,dx_3 \end{array} \leftrightarrow \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = B. \qquad \sum \frac{\partial b_i}{\partial x_i}$$

MAR 6.

Thm: $\exists!$ linear $d: \Omega^k \to \Omega^{k+1}$ s.t. 1: $f \in \Omega^0 \Rightarrow (df)(\xi) = D_\xi f = \sum_i \frac{\partial f}{\partial x_i} dx_i$

2. $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$. 3. $d^2 = 0$

pf: 1-3. imply $d(\sum a_I dx_I) = \sum_{J,I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$ proves uniqueness.

$$\begin{array}{ccccccc} \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \rightarrow & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\ \{f\} & & \{a_1dx_1 + a_2dx_2 + a_3dx_3\} & & \{b_1 dx_2 \wedge dx_3 + \text{cyc perms}\} & & \{c\,dx_1 \wedge dx_2 \wedge dx_3\} \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \{f\} & \xrightarrow[\nabla]{f \mapsto \text{grad}(f)} & \left\{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right\} & \xrightarrow[\nabla \times A]{A \mapsto \text{curl}(A)} & \left\{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right\} & \xrightarrow[\nabla \cdot B]{B \mapsto \text{div} B} & \{c\} \\ f & \longmapsto & \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{pmatrix} & \mapsto & \left\{\begin{array}{c} \frac{\partial a_2}{\partial x_3} - \frac{\partial a_1}{\partial x_2} \\ \cdots \end{array}\right\} = B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} & \longrightarrow & \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3} \end{array}$$

$0 \mapsto 1$: If $f: \mathbb{R}^n \to \mathbb{R}$. grad F = direction in which F increases the fastest.

pf: $D_{(x,v)} f = \sum \frac{\partial f}{\partial x_i}(x) v_i = \langle \text{grad } f, v \rangle$
$= \sum \left(\frac{\partial f}{\partial x_i} dx_i\right)(v) =$

where grad f maximal if v points in their direction of grad f.

$1 \mapsto 2$: $d(a_1 dx_1 + a_2 dx_2 + a_3 dx_3) = \frac{\partial a_1}{\partial x_1} \cancel{dx_1 \wedge dx_1} + \frac{\partial a_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial a_1}{\partial x_3} dx_3 \wedge dx_1 + \cdots$ (6 more).
$= -\frac{\partial a_1}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial a_1}{\partial x_3} dx_3 \wedge dx_1 + \cdots$ (4 more)

$$\begin{pmatrix} \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \end{pmatrix} \begin{array}{l} dx_2 \wedge dx_3 \\ dx_3 \wedge dx_1 \\ dx_1 \wedge dx_2 \end{array}$$

curl measures rotation of body immersed in a flow of water.

$\uparrow$ $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ part of rotation around z

E.g. $\boxed{\circlearrowleft}$ in $\mathbb{R}^2$, rotate? 1. flow left < flow right. $\frac{\partial a_2}{\partial x_1}$ "$\rightarrow$" flow in x direction
2. $-\frac{\partial a_1}{\partial x_2}$ in y direction

$2 \mapsto 3$. creation / annihilation of flow. $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mapsto \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}$ for real water in $\mathbb{R}^3$.

$\frac{\partial b_1}{\partial x_1} < 0$: $\frac{\partial b_1}{\partial x_1} > 0$ $\operatorname{div}(B) = 0$. B:

measures water added to the system.

$d^2 = 0$ curl $(-\operatorname{grad} F) = 0$. $\operatorname{div}(\operatorname{curl}(A)) = 0$.

Claim: what remains for pf of thm. If for $\omega = \sum a_I dx_I \in \Omega^k(\mathbb{R}^n)$, we set $d\omega = \sum_{j,I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$.
then d satisfies 1-3. a_I: 0-form. dx_j 1-form
dx_I. k-form.

$= \sum_{j,I} dx_j \wedge \frac{\partial a_I}{\partial x_j} \cdot dx_I = \sum_j dx_j \wedge \frac{\partial \omega}{\partial x_j} = d\omega$.

Mar 8.

Thm: $\exists!$ linear $d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ s.t. 1) $f \in \Omega^0 \Rightarrow (df)(\xi) = D_\xi f$; so $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$

2). $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg} \omega \wedge d\eta$. 3. $d^2 = 0$. "exterior derivative"

pp 1-3 imply: $d(\sum a_I dx_I) = \sum_{j=1}^n \sum_I \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I \overset{\text{imply}}{=} \sum_{j=1}^n dx_j \wedge \frac{\partial \omega}{\partial x_j}$

" $\sum \frac{\partial a_I}{\partial x_j} dx_I$.

pf: (of existence) Set $d\omega = \sum dx_j \wedge \frac{\partial \omega}{\partial x_j}$ verify 1-3.

1. $f \in \Omega^0$ $df = \sum dx_j \cdot \frac{\partial f}{\partial x_j} = \sum_j \frac{\partial f}{\partial x_j} dx_j$

To compare with $df(\xi) = D_\xi f$ eval both sides on $\xi(x, e_i)$

2. $d(\omega \wedge \eta) = d(a dx_I \wedge b dx_J) = d(a \cdot b\, dx_I \wedge dx_J) = \sum_{j=1}^n dx_j \wedge \frac{\partial(ab)}{\partial x_j} dx_I \wedge dx_J$

by linearity $\eta = b dx_J$ $= \sum dx_j \frac{\partial a}{\partial x_j} dx^I b dx_J + (-1)^{\deg \omega} \sum a dx_I\, dx_j \frac{\partial b}{\partial x_j} \wedge dx_J = (d\omega) \wedge \eta + (-1)^{\deg \omega} \cdot$

3. $dd\omega = 0$. ~~wlog $\omega = a dx_I$.~~

$d(d\omega) = d(\sum_j dx_j \wedge \frac{\partial \omega}{\partial x_j}) = \sum_k dx_k \wedge \frac{\partial(\sum dx_j \wedge \frac{\partial \omega}{\partial x_j})}{\partial x_k} = \sum_k \sum_j dx_k \wedge dx_j \wedge \frac{\partial}{\partial x_k} \frac{\partial \omega}{\partial x_j}$

$\overset{j \leftrightarrow k}{=} \sum_j \sum_k dx_j\, dx_k \frac{\partial^2 \omega}{\partial x_j \partial x_k} = - \sum_k \sum_j dx_k \wedge dx_j \cdot \frac{\partial^2 \omega}{\partial x_k \partial x_j}$

$d(d\omega) = -d(d\omega) \Rightarrow d(d\omega) = 0$.

$\Phi: \mathbb{R}^n \to \mathbb{R}^m$ $\quad \Omega^k(\mathbb{R}^n) \xleftarrow{\Phi^*} \overset{\omega}{\underset{\pitchfork}{\Omega^k}}(\mathbb{R}^m)$ $\quad \Phi^*(\omega)(\xi_1 \cdots \xi_k) = \omega(\Phi_* \xi_1 \cdots \Phi_* \xi_k)$ $\quad \xi_i \in T_x \mathbb{R}^n$

Property: 1. Φ^* is linear. 2. $\Phi^*(\omega \wedge \eta) = (\Phi^* \omega) \wedge (\Phi^* \eta)$

3. $(\Phi \circ \Psi)^* = \Psi^* \circ \Phi^*$ 4. $\Phi^*(d\omega) = d(\Phi^* \omega)$ (TBD)

$\mathbb{R}^2_{r,\theta} \xrightarrow{\Phi} \mathbb{R}^2_{x,y}$ $\quad \Phi\binom{r}{\theta} = \binom{r\cos\theta}{r\sin\theta}$ $\quad \omega = \frac{x dy - y dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2_{x,y})$.

$\Phi^* \omega = \Phi^* \frac{x dy}{x^2+y^2} - \Phi^* \frac{y dx}{x^2+y^2}$ $= \Phi^*(\frac{x}{x^2+y^2}) \cdot \Phi^*(dy) - \cdots$

$= \Phi^*(\frac{x}{x^2+y^2}) \cdot d(\Phi^* y) + \cdots$

$= \frac{r\cos\theta}{r^2\cos^2\theta + r^2\sin^2\theta} d(r\sin\theta) - \frac{\sin\theta}{r} d(r\cos\theta)$

$= \frac{\cos\theta}{r}(\sin\theta\, dr + r\cos\theta\, d\theta) - \frac{\sin\theta}{r}(\cos\theta\, dr - r\sin\theta\, d\theta) = d\theta$.

Mar 10.

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$\phi: \mathbb{R}^n \to \mathbb{R}^m \Rightarrow \phi^*: \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n)$

1. ϕ^* is linear 2. $(\phi\circ\psi)^* = \psi^* \circ \phi^*$ 3. $\phi^*(\omega\wedge\eta) = \phi^*(\omega)\wedge\phi^*(\eta)$ (Thm 4) 4. $\phi^*(d\omega) = d\phi^*\omega$

Example. $\mathbb{R}^2_{r,\theta} \xrightarrow[\binom{r}{\theta}\mapsto\binom{r\cos\theta}{r\sin\theta}]{\phi} \mathbb{R}^2_{x,y}$ $\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2_{x,y})$ $d\phi^*(\omega) = d(d\theta) = 0$.

$\phi^*(d\omega) =$

$\phi^*\omega = \frac{(r\cos\theta)\,d(r\sin\theta) - r\sin\theta\, d(r\cos\theta)}{(r\cos\theta)^2 + (r\sin\theta)^2} = \#$ $d(r\cos\theta) = \frac{\partial(r\cos\theta)}{\partial r}dr + \frac{\partial(r\cos\theta)}{\partial\theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta$

$d(r\sin\theta) = \sin\theta\, dr + r\cos\theta\, d\theta$ $\# = \frac{r^2(\cos^2\theta + \sin^2\theta)d\theta}{r^2} = d\theta$ so $\phi^*\omega$ is exact.

$d\omega = d\left(\frac{x\,dy - y\,dx}{x^2+y^2}\right) = dx\wedge\left(\frac{y^2-x^2}{(x^2+y^2)^2}dy + \cancel{\text{miss } dx}\right) + dy\wedge\left(\cancel{\text{miss } dy} + \frac{y^2-x^2}{(x^2+y^2)^2}dx\right)$

$d\omega = \sum dx_j \wedge \frac{\partial\omega}{\partial x_j} = dx\wedge\frac{\partial\omega}{\partial x} + dy\wedge\frac{\partial\omega}{\partial y}$

$\partial\left(\frac{x}{x^2+y^2}\right)/\partial x = \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\partial\left(\frac{-y}{x^2+y^2}\right)/\partial y = -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$= \frac{y^2-x^2}{(x^2+y^2)^2}(dx\wedge dy + dy\wedge dx) = 0.$

pf (thm 4). If $\omega = f \in \Omega^0(\mathbb{R}^m)$ & $\xi \in T_x\mathbb{R}^n$, $\xi = (x, v)$.

$(d\phi^* f)(\xi) = D_\xi(\phi^* f) = D(f\circ\phi)_x \cdot v$ ← chain rule →

$(\phi^* df)(\xi) = (df)(\phi_*\xi) = D_{\phi_*\xi} f = D_{(\phi(x), D\phi_x v)} f = Df_{\phi(x)}\cdot D\phi_x \cdot v$.

Both $\phi^*(d\omega)$ & $d\phi^*\omega$ are linear in ω. So enough to consider $\omega = a\cdot dx_I = a\cdot dx_2\wedge dx_5\wedge dx_7$

$d\phi^*\omega = d(\phi^*(a)\cdot\phi^*(dx_2)\wedge\phi^*(dx_5)\wedge\phi^*(dx_7)) = d(\phi^*(a)\cdot d\phi^*(x_2)\wedge d\phi^*(x_5)\wedge d\phi^* x_7)$

$= d(\phi^* a)\wedge d\phi^* x_2\wedge d\phi^* x_5\wedge d\phi^* x_7 + 0$

$\phi^*(d\omega) = \phi^*(da\wedge dx_2\wedge dx_5\wedge dx_7 + 0) = d\phi^* a\wedge d\phi^* x_2\wedge d\phi^* x_5\wedge d\phi^* x_7$ //

$\phi: \mathbb{R}^n_{x_j} \to \mathbb{R}^m_{y_i}$ $\omega \in \Omega^{top}(\mathbb{R}^n_{y_j}) = \Omega^n(\mathbb{R}^n_{y_i})$ $\omega = f\cdot dy_1\wedge\cdots\wedge dy_n = f\,dy_{1,2,\ldots,n} = f\,dy_{\underline{n}}$

Claim: $\phi^*(\omega) = \phi^*(f\,dy_{\underline{n}}) = \phi^* f\cdot J\phi(x)\cdot dx_{\underline{n}}$ $= \det\left(\frac{\partial\phi_i}{\partial x_j}\right) = \det(D\phi_x)$

e.g. $\phi^*(dx\wedge dy) = d(r\cos\theta)\wedge d(r\sin\theta) = \cdots \overset{\text{claim}}{=} 1\cdot r\cdot dr\wedge d\theta$.