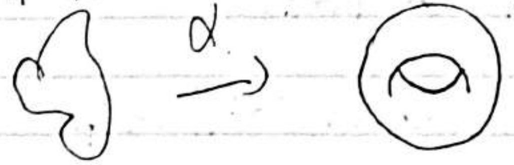
Read along: Sec 22 Mon. Jan 14^{th}

Volumn $x_1, \cdots, x_k \in \mathbb{R}^n$ in $k$dim.

$$V(x_1, \cdots, x_k) = |\det(X^T X)|^{1/2}, \quad X = (x_1 | \cdots | x_k)$$

rotation invariant. correct on $\mathbb{R}^k$

$A \subset \mathbb{R}^k$ $\qquad\qquad\qquad \subset \mathbb{R}^n$

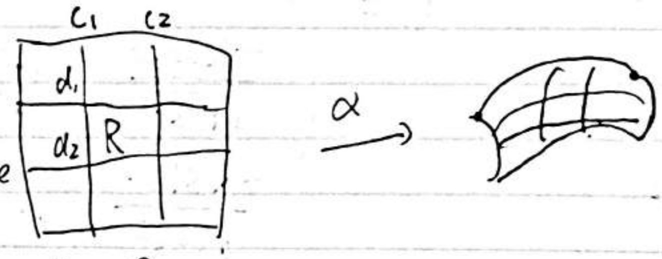


$Y = \alpha(A)$

$Vol(Y)$

Def: A "parametrized" k-manifold in $\mathbb{R}^n$ is a C^1 map $\alpha: A \to \mathbb{R}^n$, where $A \subset \mathbb{R}^k$ is some open set. $\alpha(A) = Y$. "the manifold"
α: the parametrized.

each is a tiny rectangle



$A = Q$.

$$Vol(Y) = \sum_{R \in P} Vol(\alpha R) \sim \sum_{R \in P} V\begin{pmatrix} D\alpha(c) \cdot (d_1 - c_1) e_1 \\ D\alpha(c) \cdot (d_2 - c_2) e_2 \\ \vdots \\ D\alpha(c) \cdot (d_k - c_k) e_k \end{pmatrix}$$

$R = \Pi[c_i, d_i]$

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$$

Claim: $V(x_1, \lambda x_i, \cdots, x_k) = |\lambda| V(x_1, \cdots, x_k)$. $(d_i - c_i)$ is scalar. put it out

so above $Vol(x) = \sum_{x \in P} \prod_{i=1}^{k} |d_i - c_i| \cdot V(D\alpha(c)e_1, \cdots, D\alpha(c)e_k)$

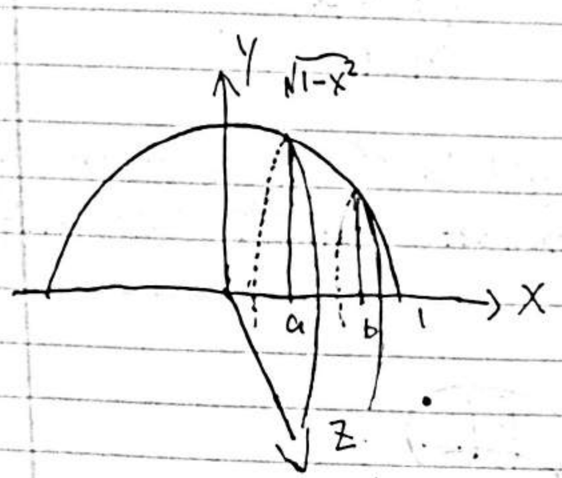
$P = \Pi[c_i, d_i]$

Remark. $X = D\alpha(c)$, $v(x_1, \cdots, x_k) = |\det X^T X|^{1/2} = v(X)$, if $X \in M_{n\times k}$

So above $= \sum_{R \in P} \mathrm{Vol}(R) \cdot v(D\alpha(c)) \sim \int_Q v(D\alpha)$ if Y is .. as above

Def : $v(Y) = v(Y, \alpha) := \int_A v(D\alpha) = \int_A |\det((D\alpha)^T D\alpha)|^{1/2}$

Example: Compute the amount of crust on a slice of a spherical loaf of bread



$A = (a,b)_x$, $[0, 2\pi]_\theta$

$$\alpha(x,\theta) = \begin{pmatrix} x \\ \sqrt{1-x^2}\cos\theta \\ \sqrt{1-x^2}\sin\theta \end{pmatrix}$$

$$D\alpha = \begin{pmatrix} 1 & 0 \\ \frac{-x\cos\theta}{\sqrt{1-x^2}} & -\sqrt{1-x^2}\sin\theta \\ \frac{-x\sin\theta}{\sqrt{1-x^2}} & \sqrt{1-x^2}\cos\theta \end{pmatrix}$$

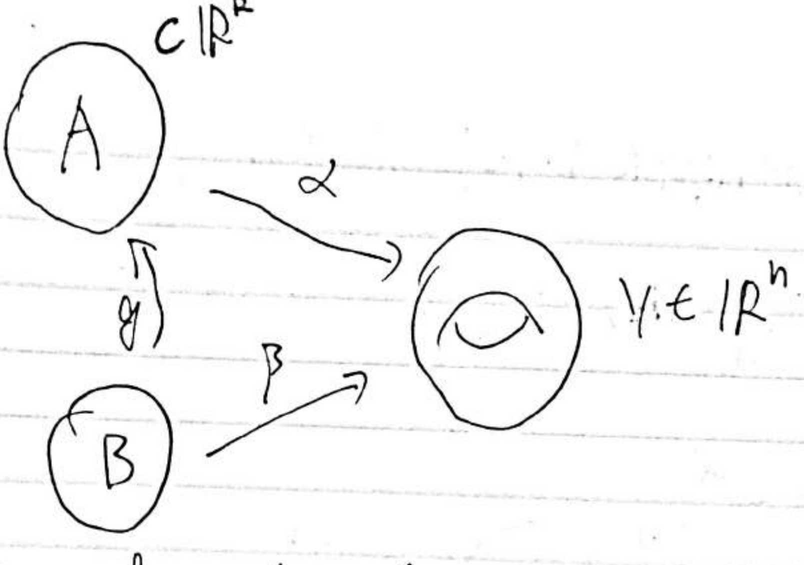
$$(D\alpha)^T D\alpha = \begin{pmatrix} \underbrace{1 + \frac{x^2}{1-x^2}}_{\frac{1}{1-x^2}} & 0 \\ 0 & 1-x^2 \end{pmatrix}$$

$X = (X|Y)$, $X^T = \begin{pmatrix} X^T \\ Y^T \end{pmatrix}$ $X^T X = \begin{pmatrix} \|X\|^2 & \langle X, Y\rangle \\ \langle Y, X\rangle & \|Y\|^2 \end{pmatrix}$

$v(D\alpha) = 1$

$\mathrm{Vol} = \int_A 1 = \mathrm{Area}(A) = 2\pi(b-a)$. $[a,b] = [-1,1]$

Surface area of a sphere is $4\pi r$.



Thm: if $\alpha : A \to \mathbb{R}^n$ is a manifold, and if $g : \underset{\mathbb{R}^k}{B} \to \underset{\mathbb{R}^k}{A}$ is diffeomorphism, then set $\beta = (\alpha \circ g)$, and then $\alpha(A) = \beta(B)$, silly
and $V(Y, \alpha) = V(Y, \beta)$.

pf: $V(Y,\beta) = \int_B V(D\beta) = \int_B V(D(\alpha \circ g))$ $\qquad D\alpha \cdot g = D_g$ at g

$= \int_B V(D\alpha(\circ g), Dg) = \int_B |\det (Dg)^T ((D\alpha \circ g)^T (D\alpha \cdot g)\ (Dg))|^{1/2}$

$= \int_B |\det Dg|\ V(D\alpha \cdot g)$

$= \int_A V(D\alpha) = V(Y, \alpha)$

Wed. Jan 18th. Manifolds in $\mathbb{R}^n$ Read along 23. 24

$$V(Y) = V(Y,\alpha) := \int_A V(D\alpha) := \int_A |\det((D\alpha)^T D\alpha)|^{1/2}$$

$$\begin{tikzcd}
\mathbb{R}^k \supset A \arrow[r, "\alpha"] & Y \subset \mathbb{R}^n \\
\mathbb{R}^k \supset B \arrow[u, "g"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

$\beta = (\alpha \circ g)$, $V(Y,\alpha) = V(Y,\beta)$

Precisely, $g: B \to A$, diffamorphism of open sets in $\mathbb{R}^k$
and $\alpha: A \to \mathbb{R}^n$ is a manifold set $\beta = \alpha \circ g$,
and then $\alpha(A) = \beta(B) = Y$, and $V(Y,\alpha) = V(Y,\beta)$

Mild generalization

$$\mathbb{R}^k \supset \boxed{A} \xrightarrow{\alpha} \subset \mathbb{R}^n = Y = \alpha(A) \xrightarrow[\text{continuous}]{F} \mathbb{R}$$

Def: $\int_Y F \, d\mathrm{vol} = \int_A (F \circ \alpha) V(D\alpha) \overset{\text{Thm}}{=\!=} \int_B (F \circ \beta) V(D\beta)$

Integral relative to k-dim volumn

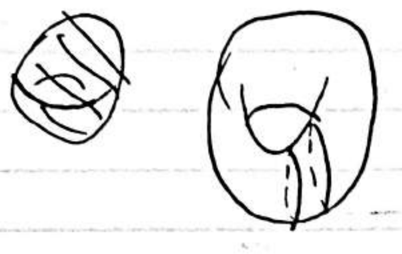
Today: $\int_{\underline{M}} d\omega = \int_{\underline{\partial M}} \omega$

M: "A nice and ~~smooth~~ sweet k-dim subset in $\mathbb{R}^n$"

Ex 0 : 0 dim in $\mathbb{R}^3$: · a point

Ex 1 : ~ $\mathbb{R}^2$ in $\mathbb{R}^3$

Ex 2 : $S^2 \subset \mathbb{R}^3$



T^2 $S^1 \times S^1$. use Σ.

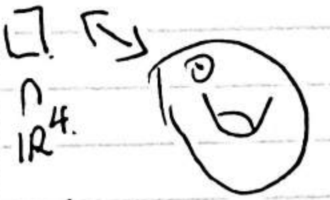
Σ_g $(g \geq 0)$ g holes in a sphere PA

Ex:

k-dim: k lin. indep. $\subset \mathbb{R}^4$

$O(n) = \{A \in M_{n \times n} : A^T A = I\} \subset \mathbb{R}^{n^2}$

$O(2) = S^1 \cup S^1 \subset \mathbb{R}^4$



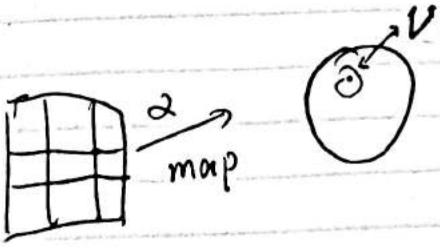
$\subset \mathbb{R}^n$

S^2

for every point. can find a subset in $\mathbb{R}^k$

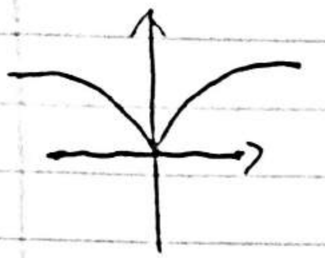
Def: A k-dim manifold (w/o boundry, of class C^r. $r \geq 1$) in $\mathbb{R}^n$. is a subset $M^k \subset \mathbb{R}^n$ Such that each Point $\in M$ has an open nbd V such that there is an open $U \subset \mathbb{R}^k$ and a C^r homemorphism $\alpha: U \to V$ whose differential $D\alpha(x)$ is of rank k, for every $x \in U$.

"coordinate patch"



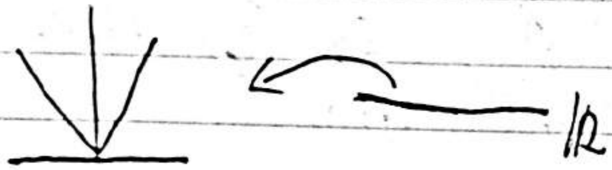
non-Examples : ① $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $t \mapsto (t^3, t^2)$

α is a homeomorphism



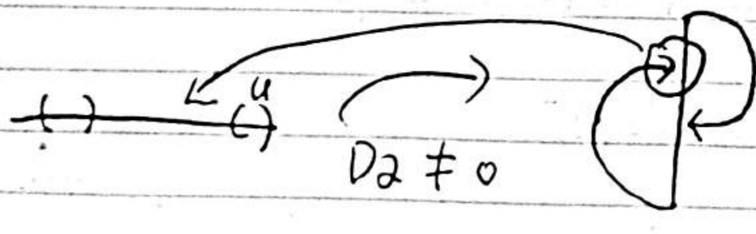
$D\alpha = (3t^2, 2t)$, not rank 1 at $t=0$

②



$(t^3, |t|^3) \longleftarrow\!\shortmid\; t$

③



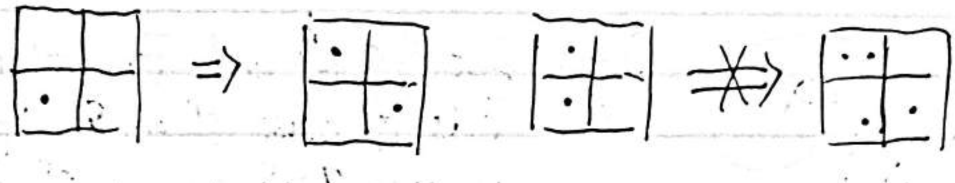
could write in coordinates

Jan. 22nd Friday ... Manifolds in $\mathbb{R}^n$

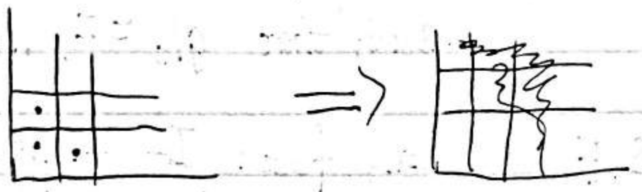
Read along: Section 23, 24

Riddle: Gal's Riddle, from 16-425

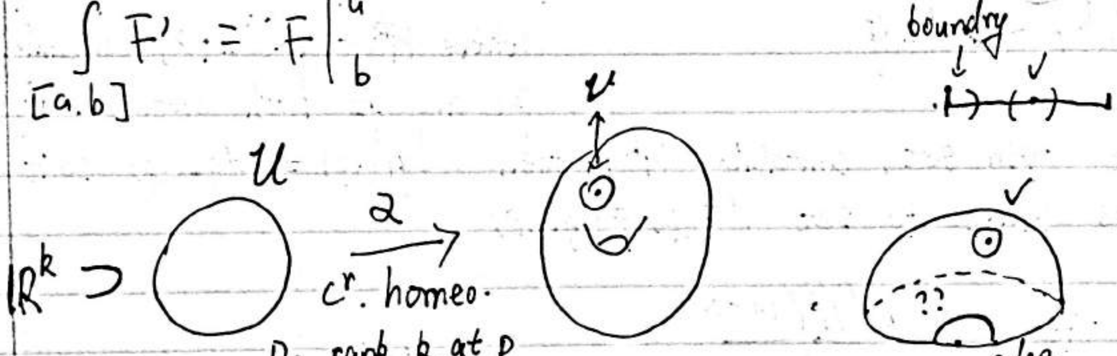
Rules:



Goal:



$$\int_{[a,b]} F' := F\Big|_b^a$$



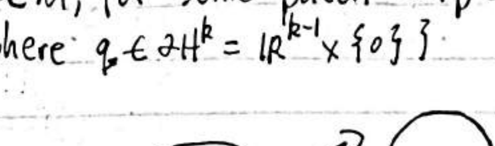
$D\alpha$ rank k at p
for each point on u

Premature Def: Let $H^k = \{x \in \mathbb{R}^k : x_k \geq 0\}$ → half of $\mathbb{R}^n$

A k-manifold (of class c^r, possibly with boundry) is a subset $M \subset \mathbb{R}^n$ such that each $p \in M$ has an open nbd (in M) such that there is an open $V \subset H^k$, and a C^r homeomorphism $\alpha : u \to v$ whose differential is of rank k, at every $x \in \mathcal{U}$.

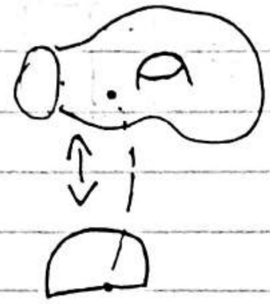
The ∂M (namely boundry, not bd$M$. Same word with different meaning and notation) of M is $\partial M \{p \in M;$ for some patch $\alpha : p = \alpha(q)$
where $q \in \partial H^k = \mathbb{R}^{k-1} \times \{0\}\}$

* α is a map, p is image ~~of~~ of q.
which at half space H^k

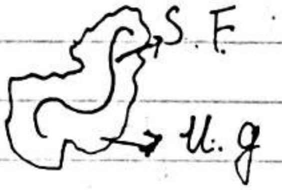


Claim: ∂M is also a manifold, without boundry, of dim $(k-1)$

Issues: 1. What does diffability mean on H^k?
2. Could it be that $\partial M = M$?
3. Why is the claim true?

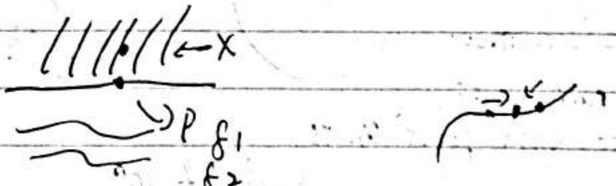


Def: Let $S \subset \mathbb{R}^k$ and $F: S \to \mathbb{R}$, we say that F is of class C^r on S, if there exists an open $\mathcal{U} \supset S$ and a C^r function $g: \mathcal{U} \to \mathbb{R}$ such that $g|_S = F$, namely $\forall x \in S$, $F(x) = g(x)$



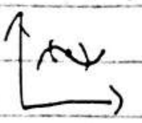
Easy fact: If $F: H^k \to \mathbb{R}^m$, then $dF(p)$ is well-defined, even for $p \in \partial H^k$, where $H^k = \mathbb{R}^{k-1} \times \{0\}$, meaning if g_1 and g_2 both extend F to some open set, containing H^k, then $(dg_1)(p) = (dg_2)(p)$, so it makes sense to set $dF(p) = dg_1(p)$

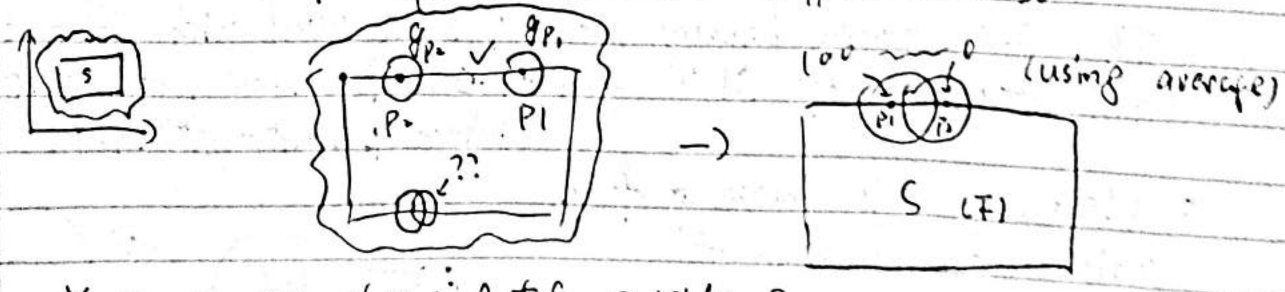
Pf: F.



know that g_1, g_2 has derivative at p. can approach from x, which is independent of choosing g_1/g_2.

Suprisingly hard fact: Differentiability on a set S is a "local property".
If $F: S \to \mathbb{R}$ has the property that for every $p \in S$, there is a nbd V_p and a diffable function $g_p: V_p \to \mathbb{R}$ such that $F|_{S \cap V_p} = g_p|_{S \cap V_p}$, then F is diffable on S. So:





$\times$ p_1, p_2 are close $\therefore$ $g_1 \neq g_2$ outside S.

Partition of Utility Lemma (Actually big theorem)

Given a collection A of open sets in $\mathbb{R}^n$, whose overall union is $A = \bigcup_{v \in A,\ \text{openset}} v$, there exist a sequence $\{\phi_i\}$ of non-negative compactly supported ϕ_i: C^∞ functions such that the following holds

[Explination: Supported ϕ: $\text{Supp}(\phi) = \overline{\{x : \phi(x) \neq 0\}}$
compactly supported : $\text{Supp}(\phi)$ is compact.]

1. $\text{Supp}(\phi) \subset A$

2. something technical. next class

3. each supported ϕ_i is contained in one $\mathcal{U} \in A$

4. $\sum \phi_i(x) = 1$