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$f(a+h) = f(a) + Df(a)h + \varphi(h)$. $\varphi \in o(h)$.

$a \in \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$, $g\circ f$ — Assumptions $a \in \mathbb{R}^n$, $f$ is diff. at $a$, $g$ is diff. at $f(a)$.

Thm. $\underbrace{D(g\circ f)(a)}_{p\times n} = \overbrace{\underbrace{Dg(f(a))}_{p\times m} \cdot \underbrace{Df(a)}_{m\times n}}^{p\times n}$

Example. $\mathbb{R}_t \xrightarrow[t \to (t,t)]{f} \mathbb{R}^2_{x,y} \xrightarrow[g]{(x,y)\to x^y} \mathbb{R}$

$(g\circ f)(t) = g(f(t)) = g(t,t) = t^t$ $\quad D(g\circ f)(t) = ((t^t)')$.

$Df(t) = \begin{pmatrix} \partial t/\partial t \\ \partial t/\partial t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = A$ $\quad Dg\begin{pmatrix} x \\ y \end{pmatrix} = (\partial g/\partial x \quad \partial g/\partial y) = (yx^{y-1} \quad x^y \log x)$

$Dg(f(t)) = (t t^{t-1} \quad t^t \log t) = B$

$\Rightarrow D(g\circ f)(t) = ((t^t)') = BA = (tt^{t-1} \quad t^t\log t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = t^t(1+\log t)$

proof $b = f(a)$. $f(a+h) = f(a) + Df(a)\cdot h + \phi(h)$. $\phi \in o(h)$.

$g(b+k) = g(b) + Dg(b)\cdot k + \gamma(k)$ $\quad \gamma \in o(k)$.

$(g\circ f)(a+h) = g(f(a+h)) = g(f(a) + Df(a)\cdot h + \phi(h))$

$= g(b) + Dg(b)\cdot(Df(a)\cdot h + \phi(h)) + \gamma(Df(a)\cdot h + \phi(h))$

$= (g\circ f)(a) + \underbrace{Dg(f(a))\cdot Df(a)}\cdot h + \underbrace{Dg(b)\cdot\phi(h) + \gamma(Df(a)\cdot h + \phi(h))}_{\lambda(h)}$.

Claim: $\lambda(h) \in o(h)$. normally $\lim_{h\to 0} \frac{\lambda(h)}{|h|} = 0$.

proof: $\frac{|Dg(b)\cdot\phi(h)|}{|h|} \le m\cdot|Dg(b)| \frac{|\phi(h)|}{|h|} \xrightarrow[h\to 0]{} 0$ $\quad * |Av| \le n\cdot|A|_{sup}\cdot|v|_{sup}$ $\quad A \in M_{m\times n}$

$\frac{|\gamma(Df(a)\cdot h + \phi(h))|}{|h|} = \frac{|\gamma(Df(a)\cdot h + \phi(h))|}{|Df(a)h + \phi(h)|} \cdot \frac{|Df(a)h + \phi(h)|}{h}$

Lemma: $\gamma: \mathbb{R}^m \to \mathbb{R}^p$ $\gamma \in o(h)$.

$\delta: \mathbb{R}^n \to \mathbb{R}^m$ $\exists C, |\delta(x)| \le C|x|$.

then $\gamma\cdot\delta \in o(x)$, that is $\frac{\gamma(\delta(x))}{|x|} \xrightarrow{x\to 0} 0$

$|\delta(h)| \le n\cdot|Df(a)|\cdot|h| + |h| = (n\|Df(a)\| + 1)|h| = C|h|$ $\square$.

Pf of lemma: $\frac{\gamma(f(x))}{|x|} = \frac{\gamma(\delta(x))}{C(x)}\cdot\frac{C|x|}{|x|}$ "prof: I owe you a proof."

Cor1. Compostion of two $C^r$ functions is $C^r$ $\quad f,g \in C^r \Rightarrow g\circ f \in C^r$

proof: (by induction on $r$).

$\lim_{h\to 0} \frac{|\gamma(h)|}{|h|} = 0$. $\gamma(0) = 0$.

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Claim. If $\gamma \in o(h)$ & $\gamma(0) = 0$. and $\delta(x) < C\cdot|x|$. for a fixed $C$ & small $x$, then $\gamma\circ\delta \in o(x)$.

*proof: $\frac{\|\gamma(\delta(x))\|}{\|x\|} = \frac{\|\gamma(\delta(x))\|}{\|\delta(x)\|}\frac{\|f(x)\|}{\|x\|}$ If $\|\delta(x)\| \ne 0$. If $\|\delta(x)\| = 0$, then $\delta(x) = 0$. then $\gamma(\delta(x)) = 0$ and the above quotient is 0.

otherwises, $\# \le$ Small $\cdot C \xrightarrow{x\to 0} 0$

Often. $\mathbb{R}^n_{x_1 \cdots x_n} \xrightarrow{f} \mathbb{R}^m_{y_1 \cdots y_m}$ where $f = \begin{pmatrix} f_1(x_1 \cdots x_n) \\ \vdots \\ f_m(x_1 \cdots x_n) \end{pmatrix}$

$\mathbb{R}^m_{y_1 \cdots y_m} \xrightarrow{g} \mathbb{R}$ where $g = g(y_1 \cdots y_m)$.

Then $\frac{\partial}{\partial x_i} g(f_1(x_1 \cdots x_n), f_2(x_1 \cdots x_n) \cdots f_m(x_1 \cdots x_n)) = ?$

Note that $D(g \circ f) = Dg \circ Df$

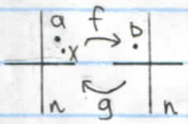
$$\left(\frac{\partial(g\circ f)}{\partial x_1} \cdots \frac{\partial(g\circ f)}{\partial x_n}\right) = \left(\frac{\partial g}{\partial y_1} \cdots \frac{\partial g}{\partial y_m}\right) \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ & \vdots & \\ \partial f_m/\partial x_1 & \cdots & \partial f_m/\partial x_n \end{pmatrix}$$

$$= \frac{\partial g}{\partial y_1}\frac{\partial f_1}{\partial x_i} + \frac{\partial g}{\partial y_2}\frac{\partial f_2}{\partial x_i} + \cdots + \frac{\partial g}{\partial y_m}\frac{\partial f_m}{\partial x_i} \quad \text{(all at } f(a))$$

$$= \frac{\partial g}{\partial y_1}\frac{\partial y_1}{\partial x_i} + \frac{\partial g}{\partial y_2}\frac{\partial y_2}{\partial x_i} + \cdots + \frac{\partial g}{\partial y_m}\frac{\partial y_m}{\partial x_i}$$

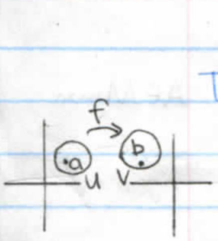
Cor 1: If $f$ & $g$ are $C^r$ then so is $g \circ f$.

proof: (induction) $r-1 \to r$. show the whole setence is diff. cont. $\Leftarrow$ show $\frac{\partial g}{\partial y_j}\frac{\partial f_j}{\partial x_i}$ is diff. cont.
equivalent to show $\frac{\partial g}{\partial y_j}$ & $\frac{\partial f_j}{\partial x_i}$ do. but it is given. $\Rightarrow$ proved.



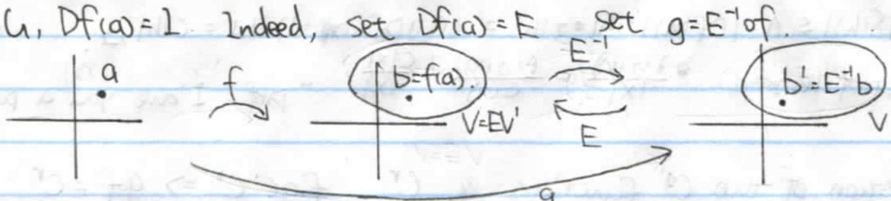
Cor 3. If $f, g : \mathbb{R}^n \to \mathbb{R}^n$, $f$ is diff. at $a$, $g$ is diff at $f(a) = b$ & near $a$, $g(f(x)) = x$.
then $Dg(b) = [Df(a)]^{-1}$

proof: Compute $D(*)$ at $a$, $(Dg)(f(a)) \cdot Df(a) = D\,Id(a) = Id$ ← a linear transformation
$$\overset{\shortparallel}{Dg(b) \cdot Df(a)}.$$
$\square$.



Thm. (the inverse function thm). If $f : \mathbb{R}^n \to \mathbb{R}^n$ diff. near $a \in \mathbb{R}^n$ and $Df(a)$ is invertible, then $f$ is invertible near $a$.
Pricisely, $\exists$ a nbd $U$ of $a$ & $V$ of $b = f(a)$ s.t. $f|_U : U \to V$ is 1-1 & onto.
Further~~m~~more, if $f$ is $C^r$, then so is the inverse $(f|_U)^{-1}$ ← f restrict to U.

WLOG, $Df(a) = I$. Indeed, set $Df(a) = E$ set $g = E^{-1} \circ f$.




$Dg(a) = E^{-1} \cdot Df(a) = E^{-1}E = I$ by IFT for $g$. find $g^{-1}$, $V$, $V'$.
Set $V = EV'$, and check that $f^{-1} = g^{-1} \circ E^{-1}$ satisfies all conditions.

Technical-Lemma. $f$ is "Jelly-rigid" near $a$, for $x, y$ near $a$, $f(y) - f(x) \sim y - x$.

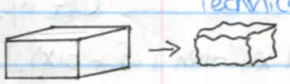


image of push/touch a cube of Jelly & a very small move of each part.

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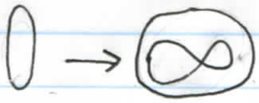
* locally inverseable, not globally: 

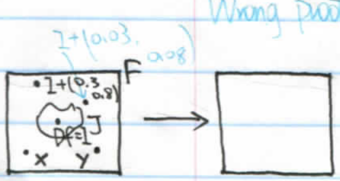
Jelly ≠ rigid.

image a spoon in Jelly, the way you can move the spoon is depend on the size of spoon

Thm: $\forall \varepsilon>0$, $\exists$ a nbd $J = J_\varepsilon = U(0,\delta)$ s.t. $\forall x,y \in J$, $\|f(y)-f(x)-(y-x)\| \le \varepsilon\|y-x\|$

Wrong proof: $y = x + \overbrace{(y-x)}^{h}$ $\quad F(y)=F(x+(y-x)) = F(x)+Df(x)\cdot(y-x)+\underbrace{\varphi(y-x)}_{\text{tiny}}$

$= F(x) + (I+\underbrace{B}_{\text{small matrix}})(y-x) + \varphi(y-x)$




$\Rightarrow F(y)-F(x)-(y-x) = B(y-x)+\varphi(y-x)$. But B can be << $\varepsilon\|y-x\|$, so does $\varphi$

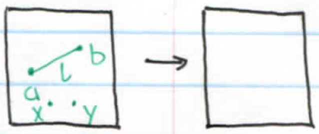
if $\varepsilon = 0.1$ $\quad \left\|\frac{\varphi(h)}{\|h\|}\right\| \le 0.01\varepsilon \quad \varphi(h) \le 0.01\varepsilon\|h\|$.

$\varphi$ is depend on x: for each x you have a different $\varphi_x$ $\quad f(a+h)=f(a)+Df(a)h+\varphi(h)$.

but we want prove the thm in every point in the "Jelly box".

Correct proof: MVT to be ~~contin~~ rescue!

$\frac{F(b)-F(a)}{b-a} = F'(c)$ where $c\in(a,b)$




Aside MVT in $\mathbb{R}$? if $f:\mathbb{R}^n\to\mathbb{R}$ is diff. on the line between a & b, then

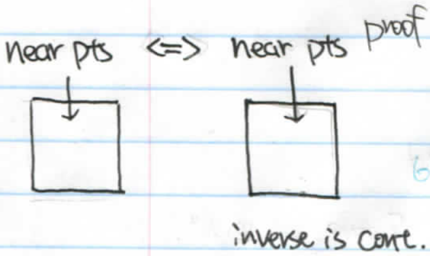
$\exists c \in l$ s.t. $f(b)-f(a) = Df(c)\cdot(b-a)$.

proof (of MVT in $\mathbb{R}^n$): Consider $g:[0,1]\to\mathbb{R}$. $\quad g(t)=f(a+t(b-a))$

by the chain rule, it is diff. so can use the 1-dim MVT here.

By 1-D. MVT, $\exists t_0\in[0,1]$ s.t. $g(1)-g(0) = g'(t_0)\cdot(1-0)$

$f(b)-f(a) \overset{*}{=} Df(c)\cdot(b-a)$. at $t_0$ $(b-a)$




proof of TC: find $c_1\cdots c_n$ between x & y s.t. $f_i(y)-f_i(x) = Df(c_i)_i(y-x)$ (ith row)

~~=I+D~~ $\quad = (I+D_i)_i$ has all entries less than $\varepsilon/n$

$= y_i - x_i + \underbrace{d_i}_{\text{small}}(y-x)$.

So $\|f_i(y)-f_i(x)-(y_i-x_i)\| \le d_i\|y-x\|$

6 steps left.

on all of J, provide J is small enough.

So $\|f(y)-f(x)-(y-x)\| \le \varepsilon\|y-x\|$