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Problem

Given $y'' + p(x)y' + q(x)y = g(x)$. find & study power series $y = \sum_{n=0}^{\infty} a_n x^n$ solutions

example

The Airy equation. $y'' = xy$. a_0, a_1 arbitrary $a_2 = 0$

$\underbrace{(n+2)(n+1)a_{n+2}}_{\text{coefficient of } x^n \text{ in } y''} = a_{n-1}$

$$n(n-1)a_n = a_{n-3} \longrightarrow a_n = \frac{a_{n-3}}{(n-1)n}$$

Solutions

$y_1(0) = 1$, $y_1'(0) = 0$: $\quad y_1 = \underbrace{1}_{a_0} \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + \cdots = 1 + \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}x^6 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}x^9$

$y_2(0) = 0$, $y_2'(0) = 1$: $\quad y_2 = x + \frac{1}{3 \cdot 4}x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}x^7 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}x^{10} + \cdots$

Fuch's Theorem

If p, q, g have power series that converge in radius R, then the solutions y also have radius of convergence at least R

$$y'' + \underbrace{\frac{1}{1-x^2}}_{R=1} y = \underbrace{\frac{1}{e^x - 2}}_{\log 2}$$

a power series solution would have radius of convergence at least $\log 2$

Theorem

$v(x) = \sum_{k=0}^{\infty} V_k x^k$ (each $V_k \in \mathbb{R}^n$)

$$A(x) = \sum_{k=0}^{\infty} A_k x^k \quad (\text{each } A_k \in M_{n\times n}(\mathbb{R}))$$

$$g(x) = \sum_{k=0}^{\infty} g_k x^k \quad g_k \in \mathbb{R}^n$$

$$v' = Av + g$$

If A & g converge to radius R. & if the power series v solves equation, then v converges at at least same R

<u>proof</u>

Equation $\Leftrightarrow$ equality of coefficient of x^k for each k

$$(k+1)V_{k+1} = g_k + \sum_{j=0}^{k} A_{k-j} \cdot V_j$$

$\|M\|$ = maximal absolute value of an entry of M ("norm")

Convergence $\Rightarrow$ $\|A_k R^k\|$ is bounded

$\Rightarrow$ $\exists$ constant α, $\|A_k R^k\| \le \alpha$

$\Rightarrow$ $\|A_k\| \le \alpha R^{-k}$

Likewise, $\exists \gamma > 0$ s.t. $\|g_k\| \le \gamma R^{-k}$

Given $r < R$ need to find η s.t. $\underline{\|V_k\| < \eta \cdot r^{-k}}$ ... $(*)$

proof by induction with undetermined hypothesis

Assume $(*)$ holds for $j \le k$. need to show it hold for $j = k+1$

$$\|V_{k+1}\| = \frac{1}{k+1} \|g_k + \sum_{j=0}^{k} A_{k-j} V_j\|$$

$$\le \frac{1}{k+1}\left(\|g_k\| + \sum_{j=0}^{k} \|A_{k-j} V_j\|\right)$$

$$\le \frac{1}{k+1}\left(\|g_k\| + n\sum_{j=0}^{k} \|A_{k-j}\| \cdot \|V_j\|\right)$$

$$\le \frac{1}{k+1}\left(\gamma R^{-k} + n\sum_{j=0}^{k} \alpha R^{j-k} \eta \cdot r^{-j}\right)$$

$$= \frac{1}{k+1}\left(\gamma R^{-k} + n r^{-k} \alpha \eta \sum_{j=0}^{k} R^{j-k} r^{k-j}\right)$$

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$$\boxed{\begin{array}{c}\text{assumption}\\ \eta > \gamma\end{array}}$$

$$= \frac{1}{k+1}\Big(\gamma R^{-k} + \alpha\eta n r^{-k} \underbrace{\sum_{j=0}^{k}\left(\frac{r}{R}\right)^{k-j}}_{\cap}\Big)$$

$$\sum_{j=0}^{\infty}\left(\frac{r}{R}\right)^{j} = \beta$$

$$\leq \frac{1}{k+1}\left(\eta r^{-k} + \alpha\beta\eta n r^{-k}\right)$$

$$= \underbrace{\frac{\eta}{k+1}}_{\text{constant}} r^{-(k+1)} \underbrace{\left(r(1+\alpha\beta n)\right)}_{\text{constant}}$$

Choose K s.t. above K $\frac{1}{K+1}(\text{const}) \leq \eta$

$$\leq \eta \cdot r^{-(k+1)}$$

$\square$

For proof to work, need η s.t. $\eta \geq \max(\gamma, \|v_k\| r^k)$ $k \leq K$