Fibonacci: How many ways to climb 'n' stairs in leaps of 1 or 2?

Sol: $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$

Generating Function

$$f = \sum_{n=0}^{\infty} F_n \cdot x^n$$

method 1: what if the generating function for climbing n stairs in 7 leaps?

$\Rightarrow$ First, find the generating function.

$$f : \{(a_1, \dots, a_n) \mid \sum a_i = n_j,\ 1 \le a_i \le 2\}$$

$$\Rightarrow \underbrace{(x+x^2)(x+x^2)\cdots(x+x^2)}_{7 \text{ leaps}}$$

$$\Rightarrow (x+x^2)^7$$

gen. fct of climbing in exactly k leaps

$$f = \sum_{n=0}^{\infty} F_n \cdot x^n = \sum_{k=0}^{\infty} (x+x^2)^k$$

so $$f = \frac{1}{1-(x+x^2)} = \frac{1}{1-x-x^2}$$

recall: $\sum_{k=0}^{\infty} y^k = \frac{1}{1-y}$

method 2: $f = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$

$xf = x + x^2 + 2x^3 + 3x^4 + \cdots$

$x^2 f = x^2 + x^3 + 2x^4 + \cdots$

$\Rightarrow f - xf - x^2 f = 1$

$(1 - x - x^2) f = 1$

$\Rightarrow f = \dfrac{1}{1 - x - x^2}$

Instead of F_n, let's find a formula for

$G_0 = 1,\ G_1 = 4,\ G_n = 5G_{n-1} - 6G_{n-2}$

$G_n = 1, 4, 14, 46, \cdots$

$\Rightarrow$ Guess $G_n = \alpha^n$

$\Rightarrow \alpha^n = 5\alpha^{n-1} - 6\alpha^{n-2}$

So $\alpha^2 = 5\alpha - 6 \Rightarrow \alpha^2 - 5\alpha + 6 = 0$,

$(\alpha_1, \alpha_2) = (2, 3) \Rightarrow 2^n$ & 3^n solve $G_n = 5 G_{n-1} - 6G_{n-2}$

$\Rightarrow \pi \cdot 2^n + 7 \cdot 3^n$ solves $G_n = 5 \cdot G_{n-1} - 6G_{n-2}$

Next, find a, b s.t. $G_n = a \cdot 2^n + b \cdot 3^n$. then

$G_0 = 1$ and $G_1 = 4$. $a + b = 1$ and $2a + 3b = 4$

Hence $a = -1,\ b = 2$

$\therefore G_n = -2^n + 2 \cdot 3^n$

Check? $n = 2$: 14. ✓

$n = 3$: 46 ✓

Guess $F_n = \lambda^n$. $F_n = F_{n-1} + F_{n-2}$

$$\Rightarrow \lambda^n = \lambda^{n-1} + \lambda^{n-2}$$

$$\Rightarrow \lambda^2 = \lambda + 1$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$