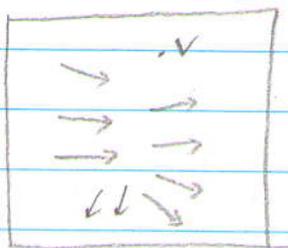


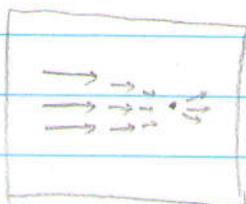
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$$F(v) \in \mathbb{R}^2, F = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix}$$

$$\dot{v} = F(v) = c_0 + A \begin{pmatrix} x \\ y \end{pmatrix} + \dots$$

$$\dot{v} = c_0 \rightsquigarrow v = c_0 t$$



near $F(v) = 0$,

$$\dot{v} = A \begin{pmatrix} x \\ y \end{pmatrix} + \dots$$

linear constant coefficient differential equation

Given $\dot{v} = Av$ near 0, what can I possibly expect?
Solutions (restrict to 2×2 matrix)

TYPE I (A has two distinct real eigenvalues)

$$A = C \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} C^{-1}, \lambda_1 \neq \lambda_2 \in \mathbb{R}$$

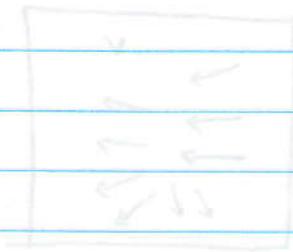
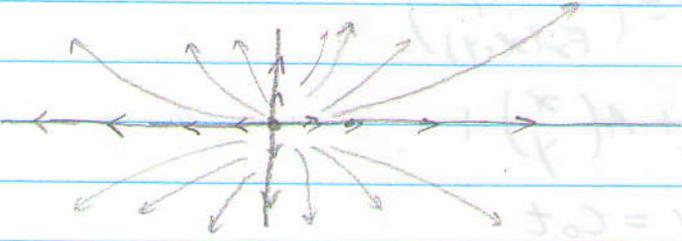
$$C = I, A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, v_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$v(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} v_0$$

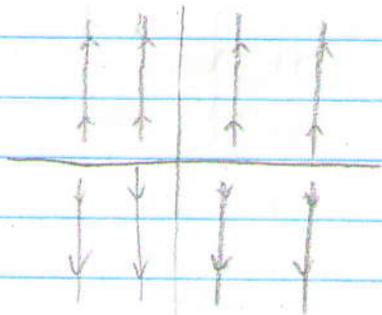
$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1 t} x_0 \\ e^{\lambda_2 t} y_0 \end{pmatrix}$$

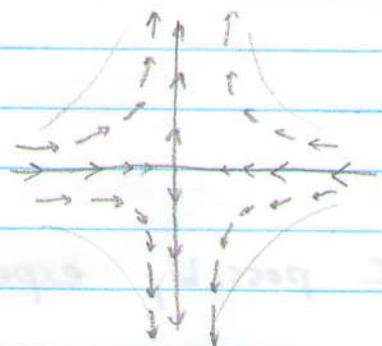
i) $0 < \lambda_1 < \lambda_2$



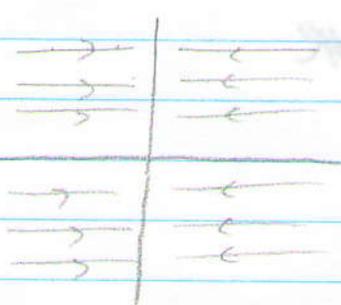
ii) $0 = \lambda_1, 0 < \lambda_2$



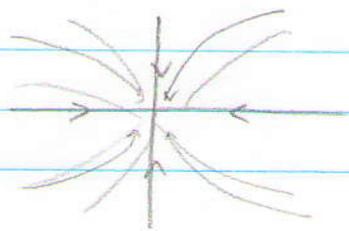
iii) $\lambda_1 < 0 < \lambda_2$



iv) $\lambda_1 < 0 = \lambda_2$



v) $\lambda_1 < \lambda_2 < 0$



matrix

1/2

$F(x) \in \mathbb{R}^2, F = (F_1(x,y), F_2(x,y))$
 $x = \begin{pmatrix} x \\ y \end{pmatrix}$
 $\dot{x} = Ax$

near $F(x) = 0$
 $\dot{x} = A(x)$

linearized equations
 $\dot{x} = Ax$

(given $\dot{x} = Ax$ near 0, what can I predict about solutions (restricted to 2D vector)

THEOREM: A has two distinct real eigenvalues

$$A = C \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} C^{-1}$$

$$C^{-1} \dot{x} = A^{-1} x = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} C^{-1} x$$

$$v(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} v(0)$$

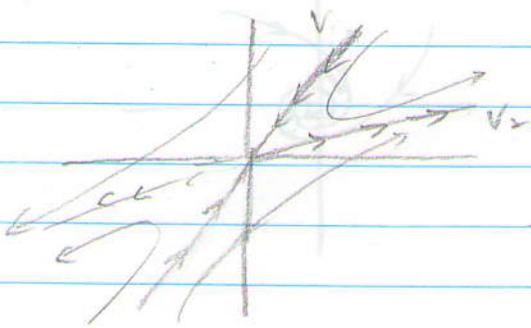
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} x_1(0) \\ e^{\lambda_2 t} x_2(0) \end{pmatrix}$$

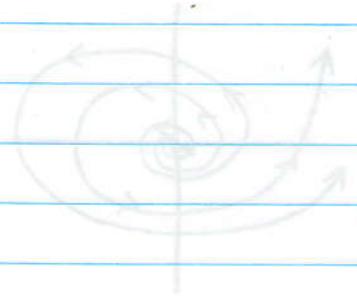
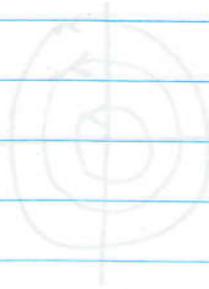
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$$AV_1 = \lambda_1 V_1 \quad \alpha > \beta$$

$$AV_2 = \lambda_2 V_2$$



$$\lambda_1 < 0 < \lambda_2$$



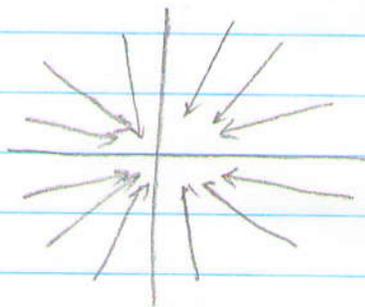
(clockwise rotation)
(counter-clockwise rotation)

rotations clockwise

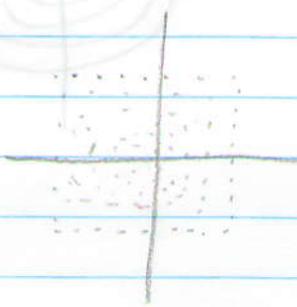
TYPE II ($\lambda_1 = \lambda_2 = \lambda$ with two linearly independent eigenvectors)

$$A = D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$$

$$\lambda < 0$$

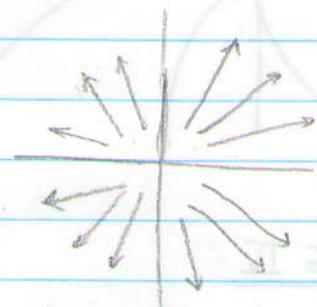


$$\lambda = 0$$



$$v = 0$$

$$\lambda > 0$$



TYPE III ($\lambda_{1,2}$ = complex conjugate pair $\lambda_{1,2} = \alpha \pm i\beta$)

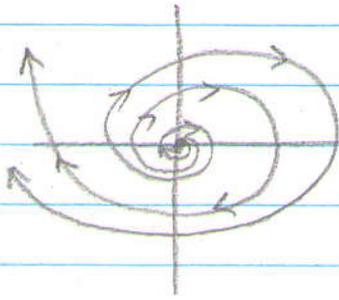
$$e^{tA} = C \begin{pmatrix} e^{\alpha t} \cos \beta t & e^{\alpha t} \sin \beta t \\ -e^{\alpha t} \sin \beta t & e^{\alpha t} \cos \beta t \end{pmatrix} C^{-1}$$

$$= C e^{\alpha t} \begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix} C^{-1}$$

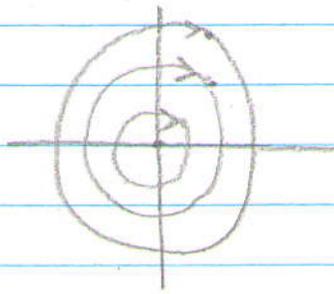
$$e^{\alpha t} \begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix}$$

$$\text{Re}(\lambda_{1,2}) = \alpha$$

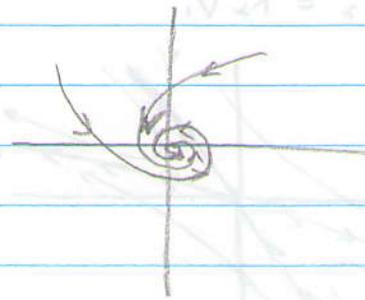
$\alpha > 0$



$\alpha = 0$



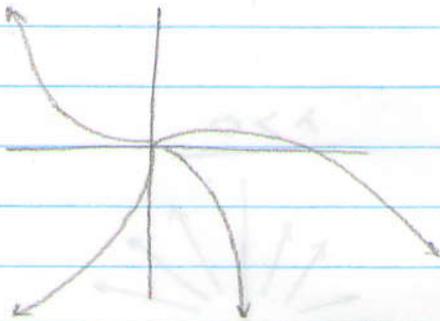
$\alpha < 0$



$$\begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix}$$

→ rotation clockwise

$\alpha \gg 0$



$\beta \gg 0$



TYPE IV

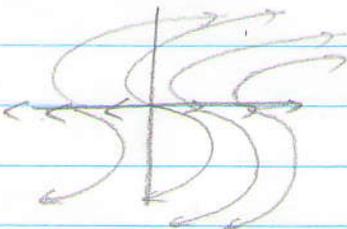
$$A \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$e^{(A)t} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$x(t) = e^{\lambda t} x_0 + t e^{\lambda t} y_0$$

$$y(t) = e^{\lambda t} y_0$$

$\lambda > 0$



$\lambda = 0$

