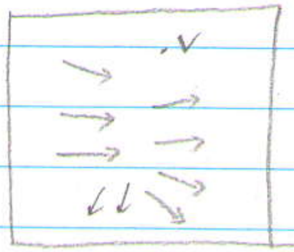
Mat267

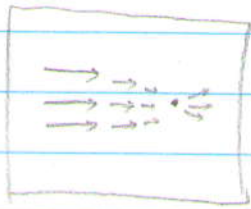
Nov. 2, 2012



$F(v) \in \mathbb{R}^2$. $F = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix}$

$\dot{v} = F(v) = C_0 + A\begin{pmatrix} x \\ y \end{pmatrix} + \cdots$

$\dot{v} = C_0 \rightsquigarrow v = C_0 t$



near $F(v) = 0$,

$$\underbrace{\dot{v} = A\begin{pmatrix} x \\ y \end{pmatrix}}_{} + \cdots$$

linear constant coefficient differential equation

Given $\dot{v} = Av$ near 0. what can I possibly expect?

Solutions (restrict to 2x2 matrix)

TYPE I (A has two distinct real eigenvalues)

$A = C\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} C^{-1}$. $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

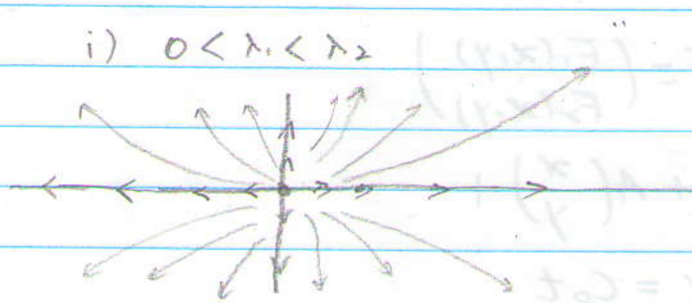
$C = I$, $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $v_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$v(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} v_0$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

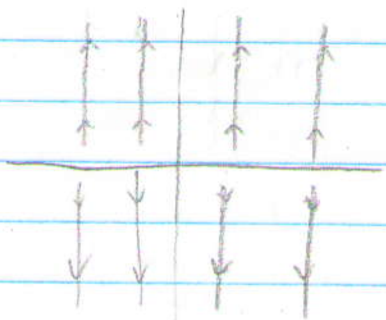
$$= \begin{pmatrix} e^{\lambda_1 t} x_0 \\ e^{\lambda_2 t} y_0 \end{pmatrix}$$

i) $0 < \lambda_1 < \lambda_2$

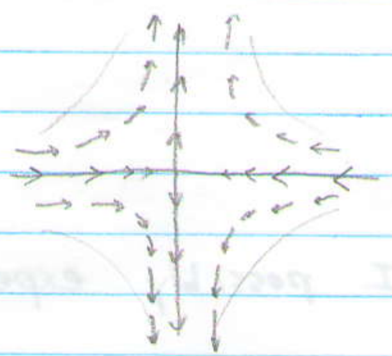


$y = x^{\frac{\lambda_2}{\lambda_1}}$

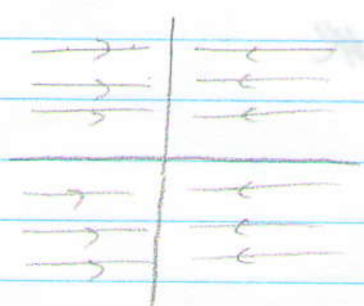
ii) $0 = \lambda_1 \quad 0 < \lambda_2$



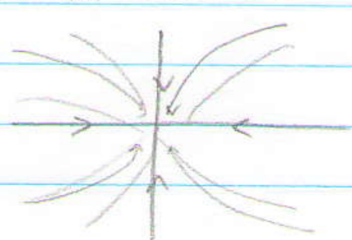
iii) $\lambda_1 < 0 < \lambda_2$



iv) $\lambda_1 < 0 = \lambda_2$



v) $\lambda_1 < \lambda_2 < 0$



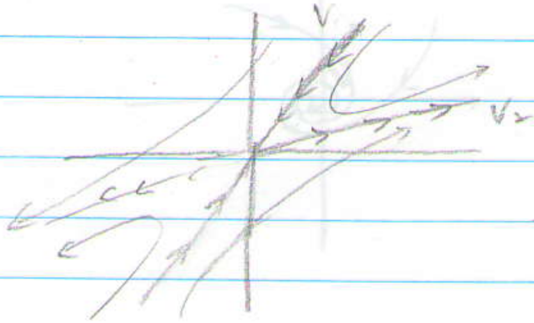
Mat267

Nov 2, 2012

2/2

$Av_1 = \lambda_1 V_1$

$Av_2 = \lambda_2 V_1$

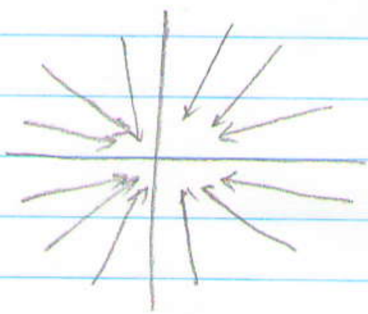


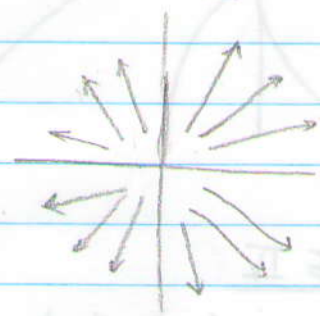
$\lambda_1 < 0 < \lambda_2$

TYPE II ($\lambda_1 = \lambda_2 = \lambda$ with two linearly independent eigenvectors)

$A = D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$

$\lambda < 0$ $\lambda = 0$ $\lambda > 0$





$V = 0$

TYPE III ($\lambda_{1,2}$ = complex conjugate pair $\lambda_{1,2} = \alpha \pm i\beta$)

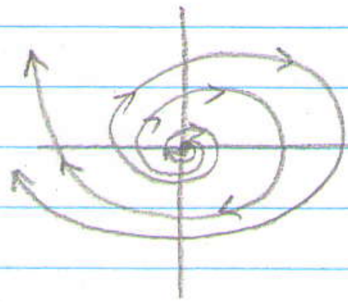
$$e^{tA} = C \begin{pmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ -e^{\alpha t}\sin\beta t & e^{\alpha t}\cos\beta t \end{pmatrix} C^{-1}$$

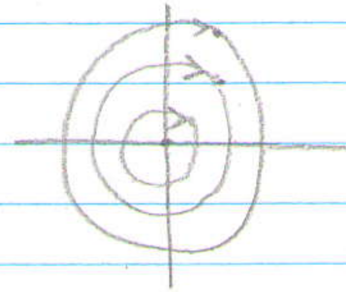
$$= C e^{\alpha t} \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix} C^{-1}$$

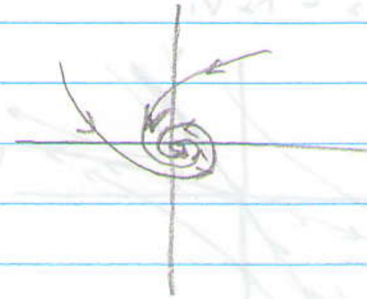
$$e^{\alpha t} \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$$

$Re(\lambda_{1,2}) = \alpha$

$\alpha > 0$ | $\alpha = 0$ | $\alpha < 0$



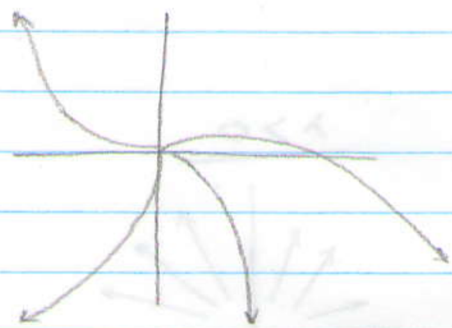




$$\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$$

$\rightarrow$ rotation clockwise

$\alpha >> 0$ | $\beta >> 0$



TYPE IV

$$A \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$x(t) = e^{\lambda t}x_0 + te^{\lambda t}y_0 \xrightarrow{\lambda = 0} x(t) = x_0 + ty_0$$

$$y(t) = e^{\lambda t}y_0 \qquad y(t) = y_0$$

$\lambda > 0$ | $\lambda = 0$