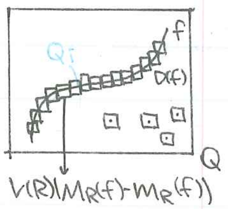
Is $Q^c = \mathbb{R}\setminus\mathbb{Q}$ of measure 0?

$\underset{\text{of mea-0}}{\mathbb{Q}} \cup (\mathbb{R}\setminus\mathbb{Q}) = \underset{\text{not mea-0}}{\mathbb{R}}$ if $\mathbb{R}\setminus\mathbb{Q}$ is of mea-0 then ~~by~~ $\mathbb{R}$ is of mea-0. $\Rightarrow\Leftarrow$

Nov. 14.

Thm. A bdd $f: Q \to \mathbb{R}$ is integrable iff its ~~co~~ discon. set is mea-0.

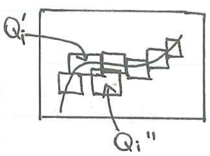
proof: Assume $|f(x)| \le M$, $\forall x \in Q$



($\Leftarrow$) Assume $D(f)$ is of mea-0. Let $\varepsilon > 0$ be given [for the riemann]. find a countable collection Q_i of rectangles s.t. $D(f) \subset \bigcup \text{int } Q_i$ & $\sum V(Q_i) < \varepsilon_1$ (ε_1 TBD)

For each $a \in Q \setminus D(f)$ find a rectangle Q_a s.t. $a \in Q_a$ & $\sup\{f(x) : x \in Q_a\} - \inf\{f(x) : x \in Q_a\}$
$= M_Q(f) - m_Q(f). < \varepsilon_2$ (TBD)

Now $\{\text{int } Q_i\} \cup \{\text{int } Q_a\}$ covers Q.



Find using compactness of Q, a finite subcover $\{\text{int } Q_1', \cdots, \text{int } Q_n'\} \subset \{\text{int } Q_i\}$ and $\{\text{int } Q_1'', \cdots, \text{int } Q_m''\} \subset \{\text{int } Q_a\}$. $Q = \bigcup_{i=1}^{n} Q_i' \cup \bigcup_{i=1}^{m} Q_i''$

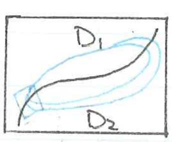
Let P be a partition of Q s.t. each Q_i'' and each Q_i' is a union of vectangles in P.

$U(f,P) - L(f,P) = \sum_{R \in P} V(R)[M_R(f) - m_R(f)] \le \underbrace{\sum_{\substack{R\in P\\ R \subset \cup Q_i'}} (\text{small})}_{A} + \underbrace{\sum_{\substack{R\in P\\ R \subset \cup Q_i''}} (\text{small})}_{B} = \#$

$A = \sum_{\substack{R\in P\\ R \subset \cup Q_i'}} V(R)(M_R(f) - m_R(f)) \le \sum_{\substack{R\in P\\ R \subset \cup Q_i'}} V(R) \cdot 2M = 2M \sum_{\substack{R\in P\\ R \subset \cup Q_i'}} V(R) = 2M \sum V(Q_i') < 2M \cdot \varepsilon_1$

$B = \sum_{\substack{R\in P\\ R \subset \cup Q_i''}} V(R)(M_R(f) - m_R(f)) < V(Q) \cdot \varepsilon_2$

$\# = A + B < 2M\varepsilon_1 + V(Q) \cdot \varepsilon_2 = \varepsilon$. change the beginning of the proof by setting $\varepsilon_1 = \varepsilon/4M$ & $\varepsilon_2 = \varepsilon/2V(Q)$. ▣.

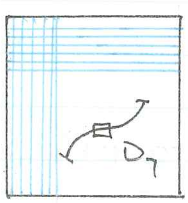


($\Rightarrow$) (sketch) Assume f is integrable.

define "osciallation of f at a" "Jump of f at a" $= V(f,a) = \inf_{R: a \in R}(M_R(f) - m_R(f))$ $D_m = \{a : V(f,a) \ge \frac{1}{m}\}$

Claim $V(f,a) > 0 \Leftrightarrow a \in D(f)$. $\Rightarrow D(f) = \bigcup_{m=1}^{\infty} D_m$

Enough to show that each D_m is of mea-0.



Find P s.t. $U(f,P) - L(f,P) < \frac{\varepsilon}{7}$

$\sum_{R\in P} V(R)(M_R(f) - m_R(f)) \ge \sum_{\substack{R \in P\\ R \cap D_7 \ne \emptyset}} V(R)(M_R(f) - m_R(f)) \ge \sum V(R) \cdot \frac{1}{7}$ mult. by 7 and get

$\Rightarrow \sum_{R \cap D_7 \ne \emptyset} V(R) < \varepsilon$. $\boxed{\Rightarrow}$

Nov. 16.

Thm. Assume $f: Q \to \mathbb{R}$ is integrable ~~i). If f=0 almost always (meaning {x∈Q: f(x)≠0}~~
i). If $f=0$ almost always (meaning $\{x\in Q: f(x)\neq 0\}$ is of mea-0) then $\int_Q F = 0$.
ii). If $f\geq 0$ and $\{x\in Q: f(x)>0\}$ is not mea-0, then $\int_Q f > 0$.

proof (of i)): $[f\neq 0] = \{x\in Q: f(x)\neq 0\}$. contains no rectangles (otherwises it would not be mea-0).
So $[f=0] = \{x\in Q: f(x)=0\}$ intersects every rectangle $R\subset Q$ $\Rightarrow \forall R$ $m_R(f)\leq 0$ $M_R(f)\geq 0$.
$\Rightarrow L(f,P) = \sum_{R\in P} V(R)\cdot m_R(f) \leq 0$. $U(f,P)\geq 0$. $L(f,P)\leq \int f \leq U(f,P)$
$\Rightarrow \int f \leq U(f,P) - L(f,P) < \varepsilon$, so $\int f < \varepsilon$, also $-\varepsilon \leq -(U-L) = L-U \leq \int f$ $\Rightarrow |\int f| \leq \varepsilon \Rightarrow \int f = 0$.

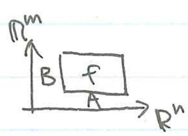
Alternatly.
$\underline{\int} f = \sup_P L(f,P) \leq 0$.
$\overline{\int} f = \inf_P U(f,P) \geq 0$
by assumption $0 \geq \underline{\int} = \overline{\int} \geq 0$.
$\Rightarrow \overline{\int} = \underline{\int} = \int = 0$.

proof of ii). $[f>0]$ not mea-0 & $D(f)$ mea-0. $\Rightarrow \exists a\in Q$ s.t. $f(a)>0$ & f is cont. at a.
$\Rightarrow \exists$ rectangle $R \ni a$ & s.t. on R, $f \geq \frac{f(a)}{2}$
$\Rightarrow m_R(f) \geq \frac{f(a)}{2}$; if P is a partition s.t. $R\in P$
then $L(f,P) = \sum_{R'} V(R')\cdot m_R(R') \geq V(R)\cdot \frac{f(a)}{2} > 0$.
$\int_Q f = \sup_P L(f,P) \geq V(R)\frac{f(a)}{2} > 0$. □

eg: pic. in 1 dim:

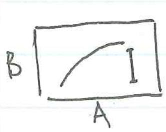
e.g. $\int_7^9 \frac{1}{1+x^2}dx$
$= \arctan 9 - \arctan 7$.

The fundamental thm of calculus: if f is cont. on $[a,b]$, then i) $F(x) = \int_a^x f$, then F' exists & $F'=f$.
ii) If g is s.t. $g'=f$, then $\int_a^b f = g(b)-g(a)$.
"~" hyped up version of $\sum_i f(t_i) \sim \sum_i g(t_i) - g(t_{i-1}) \overset{\text{telesopic summation}}{=} g(b)-g(a)$.
"~" hyped up version of "your profits over a week are equal to the sum of your daily profits"



(Fubini)
Thm. $A\subset\mathbb{R}^n$ a rectangle, $B\subset\mathbb{R}^m$ another rectangle $Q = A\times B \subset \mathbb{R}^{n+m}$
Suppose $f: Q \to \mathbb{R}$ is cont. then define for $x\in A$, $g(x) = \int_{\{x\}\times B} f$ then $\int_Q f = \int_A g = \int_A \int_B f(x,y)$.

Example. Compute $\int_{[0,1]\times[0,1]} f$ where $f(x,y) = x+y$.
$g(x) = \int_{[0,1]}(x+y)\,dy = [xy + \frac{y^2}{2}]_0^1 = x+\frac{1}{2}$. $\int_{[0,1]\times[0,1]} f = \int_{[0,1]} g = \int_0^1 (x+\frac{1}{2})dx = [\frac{x^2}{2}+\frac{x}{2}]_0^1 = 1$



Fubini's thm (full version) $Q = A\times B$ as before, Assume $f: Q\to\mathbb{R}$ is integrable
Define $l(x) = \underline{\int}_{\{x\}\times B} f$, $u(x) = \overline{\int}_{\{x\}\times B} f$, then l & u both integrable on A & $\int_A l = \int_Q F = \int_A u$.

Nov. 18.

Recall Fubini's thm.

Corollary 1: If for every $x\in A$, $f(x,-)$ is integrable over B ($\Leftrightarrow f(x,y)$ is integrable relative to $y\in B$)
then $\int_{A\times B} f = \int_{x\in A}(\int_{y\in B} f(x,y))$ e.g. true if f is cont.

Corollary 2: $\int_{A\times B} f = \int_{x\in A}\int_{y\in B} f(x,y) \overset{\text{Fubini}}{=} \int_{y\in B}\int_{x\in A} f(x,y)$, assuming all integrals in this formula exist.

e.g. $\int_0^{2\pi} d\theta \int_0^{7} dr (r\cdot\cos\theta) = \int_0^{2\pi} [\frac{r^2}{2}\cos\theta]_0^7$

Corollary 3: If all integrals exist, $Q = \prod[a_i, b_i]$ $\int_Q f = \int_{x_1\in[a_1,b_1]}\left[\int_{x_2}\left[\int_{x_3}\cdots\int_{x_n\in[a_n,b_n]} f(x_1,\dots x_n)\,dx_n\right]dx_{n-1}\right]\cdots\Big]dx_1$

proof (of Fubini):

Lemma: If $P=(P_A,P_B)$ is a partition of Q, where P_A is a partition of A & P_B is a partition of B

then $L(f,P) \overset{①}{\le} L(f,P_A) \overset{③}{\le} L(u,P_A) \overset{④}{\le} U(u,P_A) \overset{⑥}{\le} U(f,P)$

$\qquad\qquad\qquad\qquad \overset{②}{\le} U(l,P_A) \overset{⑤}{\le}$

Lemma $\Rightarrow$ Thm: f integrable $\Rightarrow$ given $\varepsilon>0$, can find $P=(P_A,P_B)$ s.t. the two extensions on the left are within ε of each other.

$\Rightarrow U(l,P_A)-L(l,P_A)<\varepsilon \Rightarrow l$ is integrable.

$|\int_Q f - \int_A l| < \varepsilon \Rightarrow \int_Q f = \int_A l$ $\square$.

proof of lemma ②-⑥ are trivial:

$u \ge l \Rightarrow$ ②, ⑤. $L \le U \Rightarrow$ ③ ④.

proof of ①: $L(l,P_A) = \sum_{R\in P_A} V(R)\, m_R(l)$

$$= \sum_{R\in P_A} V(R) \inf_{x\in R} \underline{\int}_{y\in B} f(x,y) = \sum_{R\in P_A} V(R) \inf_{x\in R} \sup_{P' \text{ of } B} L(f(x,-),P')$$

$$\ge \sum_{R\in P_A} V(R) \inf_{x\in R} L(f(x,-),P_B) = \sum_{R\in P_A} V(R) \inf_{x\in R}\Big(\sum_{R'\in P_B} V(R') \inf_{y\in R'} f(x,y)\Big) = \#.$$

Aside: e.g. $\inf_x f(x)+g(x) \ge \underbrace{\inf(f(x))}_{a} + \underbrace{\inf(g(x))}_{b}$.

proof: $\inf_x f(x)+g(x) \ge \inf_x a+b = a+b$. $\square$.

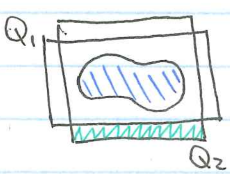
$$\# \ge \sum_{R\in P_A} V(R) \sum_{R'\in P_B} V(R') \inf_{x\in R} \inf_{y\in R'} f(x,y) = \sum_{R\in P_A}\sum_{R'\in P_B} V(R)V(R') \cdot \inf_{(x,y)\in R\times R'} f(x,y)$$

$$= \sum_{S\in P} V(S)\, m_S(f). = L(f,P) \Rightarrow ①. \quad \square.$$

⑥ is the same. $\square$.

Nov. 21

$\int_Q F \sim \int_S F$



Def. If S is a bdd subset of $\mathbb{R}^n$, define $\int_S f = \int_Q f 1_S$ where $1_S(x) = \begin{cases} 1 & x\in S \\ 0 & x\notin S\end{cases}$ "the indicator fnctn of S"

where Q is some rectangle containing S.

NTS: Def holds not depend on Q. pf. is almost empty.

Thm 1. f,g integrable over S, $a,b\in\mathbb{R}$, then $\int_S af+bg = a\int_S f + b\int_S g$.

Thm 2. f,g integrable $f\le g \Rightarrow \int_S f \le \int_S g$. $\square$

Thm 3. f integrable $|\int_S f| \le \int_S |f|$

Thm 4 If $f\ge 0$ & integrable, $T\subset S$, then $\int_T f \le \int_S f$

proof of 1: If $a,b\ge 0$, then $m_R(af+bg) \ge m_R(af) + m_R(bg) = a\,m_R(f) + b\,m_R(g)$

$M_R(af+bg) \le M_R(af) + M_R(bg) = a M_R(f) + b M_R(g)$

$\Rightarrow L(af+bg,P) \ge aL(f,P) + bL(g,P)$ $\quad U(af+bg,P) \le aU(f,P)+bL(g,P)$

$\Rightarrow U(af+bg) - L(af+bg,P)$ is small ...

Lemma: $\int(-f) = -\int f$.

Proof: $m_R(-f) = -M_R(f) \rightarrow L(-f,P) = -U(f,P)$ ... then we can go futher with arbitrary numbers.