More examples

1. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Combinatorial proof?

How many binary seq's of length n are there?

Ans 1: 2^n

Ans 2: Pick $0 \le k \le n$ & place k 1s, rest being 0's

$$\sum_{k=0}^{n} \binom{n}{k}$$

Another proof?

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\underset{\text{take } a=1,\, b=1}{\Longrightarrow} \quad (1+1)^n = \sum \binom{n}{k} 1^k 1^{n-k}$$

2. $$\binom{n}{m} \cdot \binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

Problem

Given n citizens, choose m MP within these, choose k cabinet members.

$\binom{n}{m} \cdot \binom{m}{k}$: obvious

$\binom{n}{k}$: first choose cabinets & $\binom{n-k}{m-k}$: choose the rest.

$$\underbrace{\frac{n!}{m!\,(n-m)!}\cdot\frac{m!}{k!\,(m-k)!}}_{\displaystyle \| \atop \displaystyle \frac{n!}{(n-m)!\,k!\,(m-k)!}} = \frac{n!}{k!\,(n-k)!}\cdot\frac{(n-k)!}{(m-k)!\,((n-k)-(m-k))!}$$

$$\frac{n!}{(n-m)!\,k!\,(m-k)!} = \binom{n}{n-m,\ k,\ m-k}$$

3. $$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Ans 1: Choose $k+1$ of $n+1$

$$\binom{n+1}{k+1}$$

Ans 2: First pick the largest

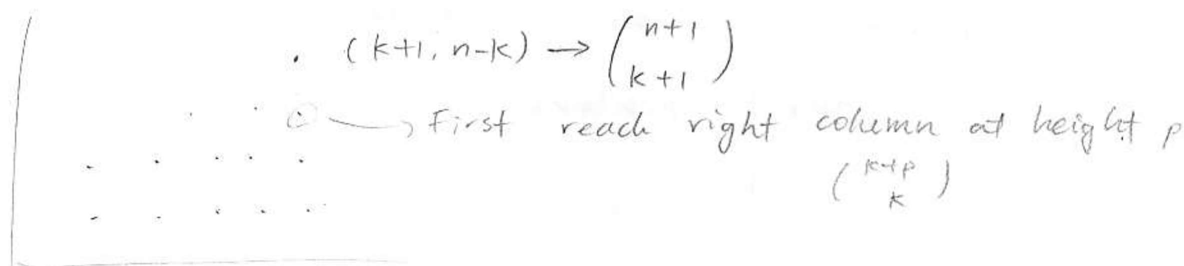
Call it p. then choose k of the remaining $p-1$.

$$\sum_{p=k+1}^{n+1}\binom{p-1}{k} = \sum_{q=k}^{n}\binom{q}{k}$$

Ans 3: Pick the largest. call it $p+1$

$$\sum_{p=k}^{n}\binom{p}{k}$$

Ans 4:



Pick the largest. call it $p+1$

$$\sum_{p=k}^{n}\binom{p}{k}$$

$$\Rightarrow \binom{n+1}{k+1} = \sum_{p=0}^{n-k}\binom{k+p}{k}$$

4. $\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$

5. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

Prob: to choose n kids out of n boys & n girls?

Ans 1: $\binom{2n}{n}$

Ans 2: For each k, choose k of the girls & skip k of the boys.

$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{k}$

6. $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

m girls

n boys

Choose r in the total.