Thm: There's a unique $v : (\mathbb{R}^n)^k \longrightarrow \mathbb{R}_{\geq 0}$ s.t.

(1) if $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an orthogonal trans & $x_i \in \mathbb{R}^n$

$v(h(x_1), \dots, h(x_k)) = v(x_1, \dots, x_k)$

(2) if $x_i \in \mathbb{R}^k \times \{0\}$. so $x_i = \begin{pmatrix} y_i \\ 0 \end{pmatrix}$ with $y_i \in \mathbb{R}^k$, then

$v(x_1, \dots, x_k) = |\det(y_1 | \dots | y_k)|$

Furthermore,

(3) $v(x_1, \dots x_k) = 0 \iff \{x_i\}$ is dependent

(4) if $X = (x_1 | \dots | x_k) \in M_{n \times k}$. then $v(x_1, \dots, x_k) = |\det(X^T X)|^{1/2}$ (5)

pf: We want to show (1) and (3) above.

Determine $v(x_1, \dots x_k)$

Let $W = \text{span}(x_1, \dots, x_k)$.

Find an o.n. basis $(f_i)_{i=1}^{l}$ of W where $l \leq k$.

Extend to a o.n. basis $(f_i)_{i=1}^{n}$ of $\mathbb{R}^n$

$$\begin{array}{ccccc} f_1 & \cdots & f_l & & f_n \\ \downarrow & & \downarrow & & \downarrow \\ e_1 & & e_l & & e_n \end{array}$$

Let $A = (f_1 | f_2 | \dots | f_n)$

So $A e_i = f_i$

A is orthogonal. so it is invertible with o.n inverse

Let h be the lin. trans. represented by A^{-1}

It is orthogonal and $h(f_i) = e_i$.

So $h(f_1), \dots, h(f_l) \in \mathbb{R}^l \subset \mathbb{R}^k$

Now, $V(x_1, \dots, x_k) \overset{(1)}{=} V(h(x_1), \dots, h(x_k))$

But each $x_i \in W$ & $h(f_i)_{i=1}^{l} \in \mathbb{R}^k \Rightarrow h(W) \subset \mathbb{R}$

So $h(x_i) \in \mathbb{R}^k$ and the RHS is determined by (2)

For existence, NTS $(5) \to (1),(2)$

1. Sps h is orthogonal meaning $h(x) = Ax$ where $A^TA = I$

$$\begin{aligned} V(h(x_1), \dots, h(x_k)) &= V(Ax_1, \dots, Ax_k) \\ &= |\det(X_h^T X_h)|^{1/2} \\ &= |\det(X^T A^T A X)|^{1/2} \\ &= |\det(X^T X)|^{1/2} \\ &= V(x_1, \dots, x_k) \end{aligned}$$

$$\boxed{\begin{aligned} X_h &= (Ax_1 | \cdots | Ax_k) \\ &= A \cdot (x_1 \cdots x_k) \\ &= AX \end{aligned}}$$

2. Sps $x_i = \begin{array}{c} \left(\begin{matrix} y_i \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right) \begin{matrix} \}k \\ \}n-k \end{matrix} \end{array}$

$$\begin{aligned} X &= (x_1 | \cdots | x_k) \\ &= \left(\begin{matrix} y_1 \\ 0 \end{matrix} \middle| \begin{matrix} y_2 \\ 0 \end{matrix} \middle| \cdots \middle| \begin{matrix} y_k \\ 0 \end{matrix} \right) \\ &= \left(\begin{matrix} \overbrace{y_1 | y_2 | \cdots | y_k}^{k} \\ \hline 0 \end{matrix} \right) \begin{matrix} \}k \\ \}n-k \end{matrix} \\ &= \begin{pmatrix} Y \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} V(x_1, \dots, x_k) &= |\det(X^TX)|^{1/2} \\ &= |\det((Y^T \ 0)\begin{pmatrix} Y \\ 0 \end{pmatrix})|^{1/2} \\ &= |\det(Y^TY)|^{1/2} \\ &= |\det(Y^2)|^{1/2} \\ &= |\det Y| \end{aligned}$$

Pf of (3): ($\Leftarrow$) $\{x_i\}$ dep $\Rightarrow \exists a \neq 0$ s.t. $X\begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} = 0$

$$\Rightarrow X^T X a = 0$$

$$\Rightarrow X^T X \text{ is not invertible}$$

$$\Rightarrow V = |\det(X^T X)|^{1/2} = 0$$

$$\Rightarrow V(x_1 \cdots x_k) = 0$$

($\Rightarrow$) $V(x_1, \dots, x_k) = 0 \Rightarrow \det(X^T X) = 0$

$$\Rightarrow \exists a \neq 0 \text{ s.t. } X^T X a = 0$$

$$\Rightarrow a^T X^T X a = 0$$

$$\Rightarrow (Xa)^T Xa = 0$$

$$\Rightarrow \|Xa\|^2 = 0$$

$\Rightarrow Xa = 0$. So columns of X are dep.

e.g: 2 C 3 : $x, y \in \mathbb{R}^3$

$$\boxed{\begin{array}{l} X = (x \mid y) \\ X^T = \begin{pmatrix} x \\ y \end{pmatrix} \end{array}}$$

$$V(x,y) = |\det X^T X|^{1/2}$$

$$= \left|\det \begin{pmatrix} \|x\|^2 & \langle x,y\rangle \\ \langle y,x\rangle & \|y\|^2 \end{pmatrix}\right|^{1/2}$$

$$= \Big|\|x\|^2\|y\|^2 - \underbrace{\langle x,y\rangle^2}_{\|x\|^2\|y\|^2\cos^2\theta}\Big|^{1/2}$$

$$= \left|\|x\|^2\|y\|^2(1-\cos^2\theta)\right|^{1/2}$$

$$= \|x\|\|y\||\sin\theta|$$