Mat240 assignment 2

1.1--3 (a) Find the equation of the plane containing the following points in space.
A(2, -5,-1), B(0,4,6), C(-3,7,1)

Vector from point A to point B is **u** = (0,4,6)-(2, -5,-1) = <-2, 9, 7>

vector from point A to point C is **v** = (-3,7,1)-(2, -5,-1) = <-5, 12, 2>

Know one point A(2, -5,-1) and two vectors **u**= <-2, 9, 7> and **v**= <-5, 12, 2> in the plane.

x= (2, -5,-1)+ su+t v

= (2, -5,-1) + s <-2, 9, 7> + t <-5, 12, 2> where $s, t \in \mathbb{R}$.

1.2--1. True or false.

In any vector space, ax=bx implies that a=b. (F) *not so when* ***x*=0**

In any vector space, ax=ay implies that x=y. (F) *not so when a=0*

18. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1,a_2), (b_1,b_2) \in V$ and $c \in \mathbb{R}$, define

$(a_1, a_2) + (b_1, b_2) = (a_1+2b_1, a_2+3b_2)$ and $c(a_1,a_2) = (ca_1,ca_2)$.

Is V a vector space over R with these operations?

<u>Solution</u>: V is **not** a vector space, since "additive associativity (x+y)+z=x+(y+z)" fails.

Counter example: $((2,2)+(1,1)) + (½, ⅓) = (2+2, 2+3) + (½, ⅓) = (4,5) + (½, ⅓) = (4+1, 5+1) = (5,6)$

But $(2,2) + ((1,1)+(½, ⅓)) = (2,2) + (1+1, 1+1) = (2,2)+(2+2) = (2+2\cdot2, 2+2\cdot3) = (2+4, 2+6) = (6,8)$

21. Let V and W be vector spaces over a field F. Let Z= {(v,w): v∈V and w∈W}.

Prove that Z is a vector space over F with operations $(v_1, w_1) + (v_2,w_2) = (v_1 +v_2,\ w_1+w_2)$ and $c(v_1, w_1) = (cv_1, cw_1)$.

Proof: Generally, the first component of a vector in Z inherits vector space properties from V, while the second component of a vector in Z inherits vector space properties from W.

1) since V is a vector space, and W is a vector space,

$\forall (v_1,u_1), (v_2,u_2)\in V$, $(v_1,u_1)+(v_2,u_2)= (v_1+v_2, u_1+u_2) =(v_2,u_2)+(v_1,u_1)=(v_2+v_1, u_2+u_1)$

⇒ for the first component, $v_1+v_2 = v_2+v_1$

$\forall (x_1,w_1), (x_2,w_2)\in W$, $(x_1,w_1)+(x_2,w_2)=(x_1+x_2, w_1+w_2)=(x_2,w_2)+(x_1,w_1)=(x_2+x_1, w_2+w_1)$

⇒ for the second component, $w_1+w_2 = w_2+w_1$

By definition, $(v_1, w_1) + (v_2,w_2) = (v_1 +v_2,\ w_1+w_2)= (v_2+v_1, w_2+w_1) =(v_2,w_2) + (v_1, w_1)$

⇒ additive commutativity holds for Z.

2) since V and W are vector spaces, ⇒

$\forall (v_1,u_1), (v_2,u_2), (v_3,u_3)\in V$, $\big((v_1,u_1)+(v_2,u_2)\big) + (v_3,u_3) = (v_1,u_1)+ \big((v_2,u_2) + (v_3,u_3)\big)$

⇒ for the first component, $(v_1+v_2)+ v_3 = v_1+ (v_2+ v_3)$

$\forall (x_1,w_1), (x_2,w_2), (x_3,w_3)\in W$, $\big((x_1,w_1)+(x_2,w_2)\big) + (x_3,w_3) = (x_1,w_1)+ \big((x_2,w_2) + (x_3,w_3)\big)$

⇒ for the second component, $(w_1+w_2) + w_3 =w_1+ (w_2+ w_3)$

$\Rightarrow\big((v_1+v_2) + v_3, (w_1+w_2)+ w_3\big) = \big(v_1+ (v_2+ v_3),\ \ w_1+ (w_2+ w_3)\big)$

⇒ By definition, $\big((v_1,w_1 ) + (v_2, w_2)\big) + (v_3,w_3) = (v_1,w_1 ) + \big((v_2, w_2) + (v_3,w_3)\big)$

where $(v_1,w_1 ),\ (v_2, w_2), (v_3,w_3)\in Z$ ⇒ additive associativity holds for Z.

3) V,W are vector spaces ⇒ ∃ zero vector $\mathbf{0}_V$ for V and zero vector $\mathbf{0}_W$ for W.

⇒ the zero vector $\mathbf{0}_Z$ for Z can be formed by

taking the first component of $\mathbf{0}_V$, and the second component of $\mathbf{0}_W$.

check that $(0_V,0_W)$ is the zero vector in Z. $(v,w)+(0_V,0_W) = (v+0_V , w+0_W)= (v,w)$

4) since $\forall (v,u) \in V$, ∃ (-v,-u) such that $(v,u)+(-v,-u)=\mathbf{0}_V$ ⇒ ∀v, ∃-v s.t. $v+(-v)=0$

$\forall (x,w) \in W$, ∃ (-x,-w) such that $(x,w)+(-x,-w)=\mathbf{0}_W$ ⇒ ∀w, ∃-w s.t. $w+(-w)=0$

$\Rightarrow \forall (v,w) \in Z$, ∃ (-v,-w) such that $(v,w) + (-v,-w) = \mathbf{0}_Z$

5) $\forall (v,u) \in V$, $1(v,u)=(v,u)$ ⇒ $1\cdot v=v$

$\forall (x,w) \in W$, $1(x,w)=(x,w)$ ⇒ $1\cdot w=w$

⇒ by definition, $1(v,w)=(1\cdot v, 1\cdot w)$ where c=1, but $(1\cdot v, 1\cdot w) = (v,w)$ ⇒ $1(v,w)= (v,w)$

6) $\forall a,b\in F$, $\forall (v,u)\in V$, $a(b(v,u))=(ab)(v,u)$ and $\forall (x,w)\in W$, $a(b(x,w))=(ab)(x,w)$

$\Rightarrow a(bv)=(ab)v$ and $a(bw)=(ab)w$ (*)

$\Rightarrow \forall (v,w)\in Z$, $a(b(v,w)) \overset{def}{=} a(bv,bw)$ *where c=b*

$\overset{def}{=} (a(bv), a(bw))$ *where c=a*

$= ((ab)v, (ab)w)$ by (*)

$\overset{def}{=} (ab)(v,w)$ *where c=ab*

7) $\forall c\in F$, $\forall (v_1,u_1), (v_2,u_2)\in V$, $c((v_1,u_1)+(v_2,u_2))= c(v_1,u_1)+c(v_2,u_2) \Rightarrow c(v_1+v_2)=cv_1+cv_2$

$\forall (x_1,w_1), (x_2,w_2)\in W$, $c((x_1,w_1)+(x_2,w_2))= c(x_1,w_1)+c(x_2,w_2) \Rightarrow c(w_1+w_2)=cw_1+cw_2$

$\Rightarrow \forall (v_1,w_1), (v_2,w_2)\in Z$,

$c((v_1,w_1)+(v_2,w_2))= c(v_1+v_2)+c(w_1+w_2)= (cv_1+cv_2, cw_1+cw_2) = c(v_1,w_1)+c(v_2,w_2)$

8) $\forall a,b\in F$, $\forall (v,u)\in V$, $(a+b)(v,u)=a(v,u)+b(v,u)=(av+bv, au+bu) \Rightarrow (a+b)v= av+bv$

$\forall (x,w)\in W$, $(a+b)(x,w)=a(x,w)+b(x,w)=(ax+bx,aw+bw) \Rightarrow (a+b)w= aw+bw$

$\Rightarrow \forall (v,w)\in Z$, $(a+b)(v,w)=(av+bv, aw+bw)= a(v,w)+b(v,w)$

1.3--8. Determine whether the sets are subspaces of $R^3$ under the operations of addition and scalar multiplication defined on $R^3$.

**A**. $W_1 = \{(a_1, a_2, a_3)\in\mathbb{R}^3 : a_1=3a_2 \ \& \ a_3 = -a_2\}$

Every vector in $W_1$ is of the form $(a_1, a_2, a_3) = (3a_2, a_2, -a_2) = a_2(3, 1, -1)$ where $a_2$ is a parameter.

Geometrically, $W_1$ is a line along the vector (3, 1, -1). The sum of any two vectors in $W_1$ is also on the line; scalar multiplication will only change the length of the line segment.

When $a_2=0$, $(a_1, a_2, a_3)=(0,0,0) \Rightarrow \mathbf{0}_{\mathbb{R}^3}\in W_1$ or the line is through the origin.

$\Rightarrow$ $W_1$ is a subspace of $\mathbb{R}^3$

~~B.~~ $W_2 = \{(a_1, a_2, a_3)\in\mathbb{R}^3 : a_1 = a_3+2\}$

Every vector in $W_2$ is of the form

$(a_1, a_2, a_3)=(a_3+2, a_2, a_3)= (2,0,0) + a_2(0, 1, 0)+ a_3(1,0,1)$ where $a_2, a_3$ are parameters.

This is a plane spanned by vectors (0, 1, 0) and (1,0,1).

When $a_2=0= a_3$, $(a_1, a_2, a_3)=(2,0,0) \Rightarrow \mathbf{0}_{\mathbb{R}^3}\notin W_2 \Rightarrow$ $W_2$ is **not** a subspace of $\mathbb{R}^3$

**C**. $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \;:\; 2a_1 - 7a_2 + a_3 = 0\}$

Every vector in $W_3$ is of the form $(a_1, a_2, a_3)=(a_1, a_2, -2a_1+7a_2)= a_1(1,0,-2) + a_2(0, 1, 7)$

When $a_1=0=a_2$, $(a_1, a_2, a_3)=(0,0,0)$ $\Rightarrow$ $\mathbf{0}_{\mathbb{R}^3}\in W_3$

This is a plane through the origin $\Rightarrow$ $W_3$ is a subspace of $\mathbb{R}^3$

**D**. $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$

Every vector in $W_4$ is of the form $(a_1, a_2, a_3)=(4a_2+ a_3, a_2, a_3)= a_2(4,1,0) + a_3(1, 0, 1)$.

When $a_2=0=a_3$, $(a_1, a_2, a_3)=(0,0,0)$ $\Rightarrow$ $\mathbf{0}_{\mathbb{R}^3}\in W_4$

This is a plane through the origin. $\Rightarrow$ $W_4$ is a subspace of $\mathbb{R}^3$

E. $W_5 = \{(a_1, a_2, a_3)\in\mathbb{R}^3: a_1 + 2a_2 - 3a_3 = 1\}$

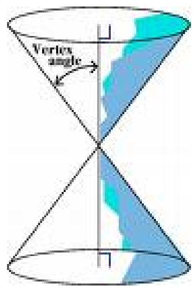
$\forall v\in W_5$, $v= (a_1, a_2, a_3) = (1-2a_2+3a_3, a_2, a_3) = (1,0,0) + a_2(-2, 1, 0) + a_3(3, 0, 1)$

*when* $a_2 = 0 = a_3$, $v=(1,0,0)$ $W_5$ represents a plane **not** through the origin

$\Rightarrow$ $W_5$ is a **not** subspace of $\mathbb{R}^3$ since $\mathbf{0}_{\mathbb{R}^3}\notin W_5$

F. $W_6 = \{(a_1, a_2, a_3) \in\mathbb{R}^3: 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

1° Notice that when $a_2= 0$, $5a_1^2 + 6a_3^2 = 0$ $\Rightarrow$ $(a_1, a_2, a_3) = (0,0,0)$ $\Rightarrow$ $\mathbf{0}_{\mathbb{R}^3}\in W_6$




*$W_6$ is a cone surface in $\mathbb{R}^3$*

2° $\forall c\in F$, $5(ca_1)^2 - 3(ca_2)^2 + 6(ca_3)^2 = c^2(5a_1^2 - 3a_2^2 + 6a_3^2) = c^2(0) = 0$

$\Rightarrow (c(a_1, a_2, a_3)) \in W_6$ $\Rightarrow$ $W_6$ is **closed under scalar multiplication**.

3° But for any two vectors $(u,v,w)$ & $(x,y,z)\in W_6$ their sum $(u,v,w)+(x,y,z) = (u+x, v+y, w+z)$

But $(a_1, a_2, a_3) = (u+x, v+y, w+z)$ does not necessarily satisfy the equation $5a_1^2 - 3a_2^2 + 6a_3^2 = 0$

$5(u+x)^2-3(v+y)^2+ 6(w+z)^2= 5(u^2+x^2+2ux) -3(v^2+y^2+2vy) + 6(w^2+z^2+2wz)$

$= (5u^2-3v^2+6w^2) + (5x^2-3y^2+6z^2)+ 2(5ux -3vy +6wz)$

$= 0 +0 +2(5ux -3vy + 6wz)$ where the cross terms are not guaranteed to get cancelled.

$\Rightarrow$ $W_6$ is **not** closed under addition. Hence, $W_6$ is **not** a subspace.

19. Let $W_1$ and $W_2$ be subspaces of a vector space V.

Prove that $W_1 \cup W_2$ is a subspace of V $\Leftrightarrow$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

First prove $\Leftarrow$:

Suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, if $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$, which is a subspace of V.

if $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_1$, which is also a subspace of V.

In either case, $W_1 \cup W_2$ is a subspace of V.

Now prove $\Rightarrow$ indirectly by a contradiction:

Assume $W_1 \cup W_2$ is a subspace of V, but $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$, now look for a contradiction.

$W_1 \not\subseteq W_2$ $\Rightarrow \exists$ a vector $u \notin W_2$ but $u \in W_1$ $\Rightarrow$ $u \in W_1 \cup W_2$

$W_2 \not\subseteq W_1$ $\Rightarrow \exists$ a vector (*) $v \notin W_1$ but $v \in W_2$ $\Rightarrow$ $v \in W_1 \cup W_2$

$W_1 \cup W_2$ is a subspace of V $\Rightarrow (u+v) \in W_1 \cup W_2$ *By property of additive closure of a subspace.*

$\Rightarrow (u+v) \in W_1$ or $(u+v) \in W_2$.

Assume that $(u+v) \in W_1$. But $u \in W_1$ $\Rightarrow$ its inverse $(-u) \in W_1$

$W_1$ is a subspace of V $\Rightarrow (-u)+(u+v) \in W_1$ *by additive closure of a subspace*

But $(-u)+(u+v) = ((-u)+u)+v$ *by additive associativity of a vector space*

$= 0 + v = v \in W_1$ $\Rightarrow$ a contradiction with assumption (*)

$\Rightarrow$ Assumption $W_2 \not\subseteq W_1$ must be wrong,

i.e. $W_2 \subseteq W_1$ holds

$\Rightarrow$ so does $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ □