Def: $h : \mathbb{R}^n \to \mathbb{R}^n$ is an "isometry" if $\forall x, y$ $d(h(x), h(y)) = d(x,y)$

Thm: h is an isometry iff it is of the form

$h(x) = p + Ax$ where $A \in M_{n\times n}$ satisfies $A^T A = I$

Already know: WLOG $h(0) = 0$, h preserves norms &

dot products. $A := (h(e_1) | h(e_2) | \cdots | h(e_n)) \in O(n)$

Claim: $h(\sum x_i e_i) = \sum x_i h(e_i)$

if true. $h\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = h(\sum x_i e_i)$

$$\overset{\text{claim}}{=} \sum x_i h(e_i)$$

$$= A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= Ax$$

Pf: Let $\Delta = h(\sum x_i e_i) - \sum x_i h(e_i)$

$\langle \Delta, h(e_j) \rangle = \langle h(\sum x_i e_i), h(e_j) \rangle - \sum x_i \langle h(e_i), h(e_j) \rangle$

$= \langle \sum x_i e_i, e_j \rangle - \sum x_i \langle e_i, e_j \rangle$

$= 0$

But $h(e_j) = Ae_j$

$0 = \langle \Delta, h(e_j) \rangle$

$= \langle \Delta, Ae_j \rangle$

$= \Delta^T A e_j \quad \forall j$

$\Rightarrow \Delta^T A = 0$

But A is invertible, so $\Delta^T = 0 \Rightarrow \Delta = 0$ $\square$

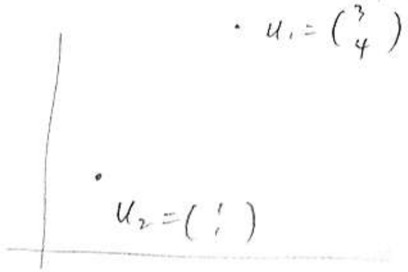
Important Aside (The Gram-Schmidt Process)

If $\{u_i\}$ is a basis of an inner-product sp. (for this class, it's okay to think $V = \mathbb{R}^n$ and $\langle a, b \rangle = a^T b$)

then there exits $\left(\begin{smallmatrix}\text{almost}\\ \text{unique}\end{smallmatrix}\right)$ orthonormal basis $\{v_i\}$ s.t.

$\forall k$, $1 \le k \le n$ in V. $\quad \text{spann}(u_i)_{i=1}^{k} = \text{spann}(v_i)_{i=1}^{k}$

e.g: $\quad u_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$



$$v_1 = \pm \frac{u_1}{\|u_1\|} = \pm \frac{\begin{pmatrix} 3 \\ 4 \end{pmatrix}}{5} = \pm \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$v_2' = u_2 - \langle u_2, v_1 \rangle v_1$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{7}{5} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$= \begin{pmatrix} 4/25 \\ -3/25 \end{pmatrix}$$

$$v_2 = \pm \frac{v_2'}{\|v_2'\|} = \frac{\begin{pmatrix} 4/25 \\ -3/25 \end{pmatrix}}{1/5} = \begin{pmatrix} 4/5 \\ -3/5 \end{pmatrix}$$

Now in general,

$$v_1' = u_1 \qquad \Rightarrow \quad v_1 = \pm \frac{v_1'}{\|v_1'\|}$$

$$v_2' = u_2 - \langle u_2, v_1 \rangle v_1 \qquad \Rightarrow \quad v_2 = \pm \frac{v_2'}{\|v_2'\|}$$

$$v_3' = u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2 \qquad \Rightarrow \quad v_3 = \pm \frac{v_3'}{\|v_3'\|}$$

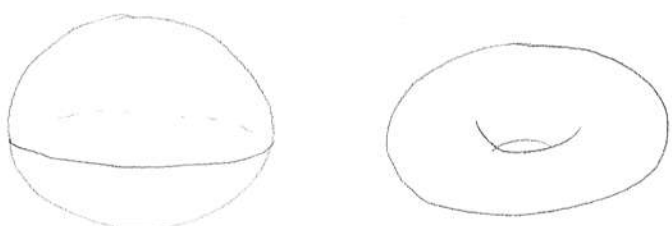
$$v_k' = u_k - \sum_{j=1}^{k-1} \langle u_k, v_j \rangle v_j \qquad \Rightarrow \quad v_k = \pm \frac{v_k'}{\|v_k'\|}$$

Claim: The process works: 1. V_i are o.n.

2. $\text{spann}(u_i)_{i=1}^{k} = \text{spann}(v_i)_{i=1}^{k}$

Pf: exercise

k-dim volume in $\mathbb{R}^n$



Q: Given $v_1, \dots, v_k$ in $\mathbb{R}^n$,

what's vol $\left(\begin{matrix}\text{parallelopiped} \\ \text{spanned by those}\end{matrix}\right) = v(v_1 \dots v_k)$

Want: 1 If $A^T A = I$. $A \in M_{n \times n}(\mathbb{R})$.

$$v(v_1 \dots v_k) = v(Av_1 \dots Av_k)$$

2. If $v_1 \dots v_k \subset \mathbb{R}^k \times \{0_{n-k}\} \subset \mathbb{R}^n$

then $v_i = \begin{pmatrix} y_i \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\Big\}{n-k}$

& $v(v_1 \dots v_k) = |\det(y_1 | y_2 | \dots | y_k)|$

thm: v exists and it's unique.