Nov 26, 2012

$(Ly := x^2y'' + xpy' + qy) = 0 \qquad p = \sum_0 p_n x^n, \ q = \sum_0 q_n x^n$

Try $x^\alpha \sum_{n=0}^{\infty} a_n x^n = \sum a_n x^{n+\alpha} \quad (a_0 = 1)$

get $F(\alpha+n)a_n = -\sum_{k=0}^{n-1} a_k [(\alpha+k)p_{n-k} + q_{n-k}] \quad (*)$

where $F(\alpha) = \alpha(\alpha-1) + p_0\alpha + q_0$ "indicial poly"

$\cancel{F(\alpha)a_0 = 0}$ take $a_0 = 1$

Use $(*)$ to get $a_n(\alpha)$

Set $\phi_\alpha = \sum a_n(\alpha) x^{n+\alpha}$ "the fundamental series of L"

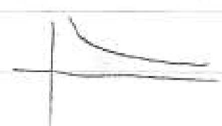
1. $L\phi_\alpha = F(\alpha) X^\alpha \quad \cdots (1)$
2. The coeff. $a_n(\alpha)$ may have poles if for some $1 \le k \le n$, $F(\alpha+k) = 0$
3. if α is such that no poles arise, then ϕ_α is convergent.

To get solutions solve $F(\alpha) = 0$. $\alpha_{1,2}$

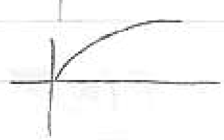
if nothing unexpected happens, solutions are $y_1 = F_{\alpha_1}, \ y_2 = F_{\alpha_2}$

$\sim x^{\frac{1}{3}}(1+\cdots) \quad \sim x^{\frac{3}{4}}(1+\cdots)$

$ay_1 + by_2$ if $b \neq 0$



otherwise



$$y'' + \left(\frac{1}{2x^2} + \frac{1}{2(1-x^2)}\right) y = 0$$

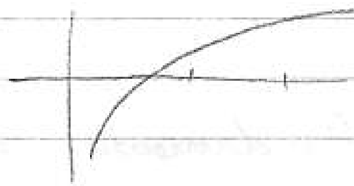
$$x^2 y'' + \underbrace{\left(\frac{1}{2} + \frac{x^2}{2(1-x^2)}\right)}_{q} y = 0$$

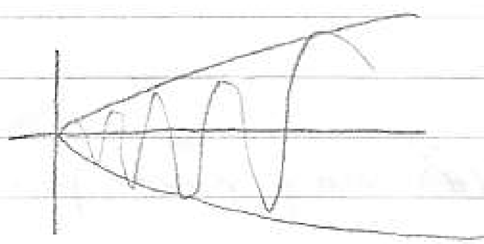
$p = 0 \qquad q \qquad q_0 = \frac{1}{2}$

$F(\alpha) = \alpha(\alpha-1) + \frac{1}{2} = 0$, $\alpha = \frac{1}{2} \pm \frac{1}{2} i$

solutions $\sim x^{\frac{1}{2}+\frac{1}{2}i} \sim x^{\frac{1}{2}} \begin{pmatrix} \cos\frac{1}{2}\log x \\ \sin\frac{1}{2}\log x \end{pmatrix}$

$$\begin{aligned} x^{\lambda} &= e^{\lambda \log x} \qquad \lambda = \alpha + i\beta \\ &= e^{(\alpha+i\beta)\log x} \\ &= e^{\alpha \log x} \cdot e^{i\beta \log x} \\ &= x^{\alpha}(\cos(\beta \log x) + i \sin(\beta \log x)) \end{aligned}$$

$\log x$ 

$x^{\frac{1}{2}}\cos(\frac{1}{2}\log x)$ 

What if F has only one root? $\alpha_1 = \alpha_2$

$$y_1 = \phi_\alpha \qquad y_2 = \frac{d}{d\alpha}\phi_\alpha \Big|_{\alpha=\alpha_1}$$

Take (1) $\frac{d}{d\alpha}$

$$L\frac{d}{d\alpha}\phi_\alpha = F'(\alpha)x^{\alpha} + F(\alpha)x^{\alpha}\log x$$

At $\alpha = \alpha_1 = \alpha_2$: $L\left(\frac{d}{d\alpha}\phi_\alpha\right)\Big|_{\alpha=\alpha_1} = 0 + 0 = 0$

Exercise : $y_2 = y_1 \log x + \sum b_n x^n$

<u>Last case</u> $\alpha_1 > \alpha_2$ $y_1 = \phi_\alpha$ but if $\alpha_1 - \alpha_2 = n \in \mathbb{N} > 0$ then ϕ_{α_2} may not make sense.

Solution : $L((\alpha - \alpha_1)\phi_\alpha) = (\alpha - \alpha_1)F(\alpha)x^{\alpha}$