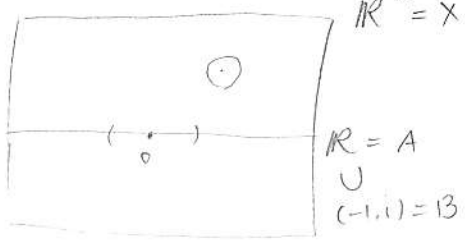
e.g.: ①

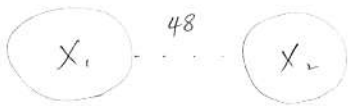


$\mathbb{R}$ closed in $\mathbb{R}^2$, but not open.

B open in A, not in X

B not closed in A, nor in X.

②

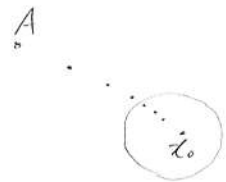


open sets : ϕ, $\{x_1, x_2\}$, $\{x_1\}$, $\{x_2\}$

closed sets : " "

③

Def: $x_0 \in X$ is called a "limit pt" of $A \subset X$ if



$$\forall \varepsilon > 0, \ (B(x_0, \varepsilon) \cap A) \setminus \{x_0\} \neq \phi$$

$$\overset{prop}{\Longleftrightarrow} \forall \varepsilon > 0, \ |B(x_0, \varepsilon) \cap A| = \infty$$

Def: The "closure" $\bar{A}$ of a set A

$$\bar{A} := A \cup \{\text{limit pts of } A\}$$

$$= A \cup lp(A)$$

e.g: ① In $\mathbb{R}$ $\overline{(0,1)} = [0,1]$

② $lp\{\overline{\tfrac{1}{n}}\} = \{0\}$

Thm: A is closed $\Leftrightarrow A = \bar{A}$

pf: same as

A^c open $\Leftrightarrow lp\, A \subset A$

$A = \bar{A} \Leftrightarrow lpA \subset A$

$(\Rightarrow)$ by contradiction, sps $x_0 \in lpA \cap A^c$

Find $\varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subset A^c$

But, then $B(x_0, \varepsilon) \cap A = \phi$

Contradicting $x_0 \in lpA$

$(\Leftarrow)$ Exercise.

Exercise: $\bar{A}$ is

1. the smallest closed set containing A
2. the intersection of all closed sets containing A

Def: Let $f: X \to Y$ $((X, d_X), (Y, d_Y))$

Let $x_0 \in X$ we say that f cts at x_0

if for every nbhd V of $f(x_0)$, there exists a nbhd U of x_0

s.t. $f(U) \subset V$.

prop $\Leftrightarrow$ $\forall \varepsilon > 0$. $\exists \delta > 0$ s.t. $d_X(x_0, x) < \delta \Rightarrow d_Y(f(x_0), f(x)) < \varepsilon$

prop. $\Leftrightarrow$ if $X = Y = \mathbb{R}$, continuity as in MAT157.

e.g: $X = C([0,1])$. $Y = \mathbb{R}$

$$\phi : \underset{\overset{\shortparallel}{X}}{C([0,1])} \longrightarrow \underset{\overset{\shortparallel}{Y}}{\mathbb{R}}$$

by $\phi(f) = f(0.3)$ when $f \in C([0,1])$

claim: ϕ cts

Thm: TFAE: for $f: X \longrightarrow Y$

1. f cts (at every pt of X)

2. . . .

⋮